Multi-loop string amplitudes and Feynman Graphs

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Based on work done with LORENZO MAGNEA and STEFANO SCIUTO (Torino) and RODOLFO RUSSO (Queen Mary University of London).

- L. Magnea, S. Playle, R. Russo, and S. Sciuto, "Multi-loop open string amplitudes and their field theory limit," *JHEP* **1309** (2013) 081, `arXiv:1305.6631 [hep-th]`.
- S. Playle, "Gauge theory effective actions from open strings," (QMUL Ph.D. thesis) 2014.
- Forthcoming:
  L. Magnea, S. Playle, R. Russo, and S. Sciuto, "Multi-loop Yang-Mills graphs from superstrings," `arXiv:14XX.XXXX [hep-th]`.

Table of contents

Presentation Outline

- It was noticed in the early days of string theory that in the limit of infinite string tension $\alpha' \to 0$, string theory reduces to Yang-Mills gauge theory coupled to gravity.

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- String amplitudes have been used to find individual Feynman graphs to compute *e.g.* one-loop Yang-Mills renormalization scattering [Di Vecchia, Lerda, Magnea, Marotta, Russo 1996a] and Φ^3 scalar scattering at 2-loops. [Di Vecchia, Lerda, Magnea, Marotta, Russo 1996b]

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  [Schwinger 1954, Ritus 1977]
- The 1 loop amplitude for open strings in a magnetic field was calculated in the mid 1980s.

  [Fradkin, Tseytlin 1985; Abouelsaood, Callan, Nappi, Yost 1988 *etc*]

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The string measure

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- g-loop string vertices can obtained by 'sewing' together *3-reggeon vertices* $\mathcal{V} \in \mathscr{H}^{*\otimes 3}_{\text{string}}$.

  [Sciuto 1969; Caneschi, Schwimmer, Veneziano 1969; Della Selva, Saito 1970; Di Vecchia, Nakayama, Petersen, Sciuto 1986]

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  [Sciuto 1969; Caneschi, Schwimmer, Veneziano 1969; Della Selva, Saito 1970; Di Vecchia, Nakayama, Petersen, Sciuto 1986]
- 'Sewing' two legs i, j means acting on the ith Hilbert space $\mathscr{H}^{*,i}_{\text{string}}$ with the BRST-invariant propagator $D(x)$ ([Di Vecchia, Frau, Lerda, Sciuto 1987]) then contracting with the dual of the string Hilbert space on leg j: $\mathscr{H}^{j}_{\text{string}}$.

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- Sewing N-reggeons leads *automatically* to an amplitude written in terms of quantities on a Riemann surface (τ_{ij}, $\omega_i(z)$, $E(z, w)$,...) expressed in the *Schottky group* formalism.

  [Lovelace 1970; Kaku, Yu 1970; Alessandrini 1971]

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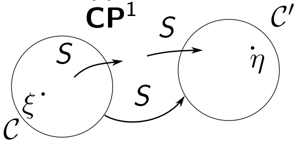
- Fixed points $\leftrightarrow$ eigenvectors; multiplier $k \leftrightarrow$ ratio of eigenvalues.
- Up to a global change of coordinates on the Riemann sphere (one taking $\eta \to 0$, $\xi \to \infty$), any Möbius map is equivalent to $z \mapsto kz$.

- Given a Möbius map, we can find circles $\mathcal{C}$ and $\mathcal{C}' = S(\mathcal{C})$ around ξ and η, such that inside of $\mathcal{C}$ is mapped to outside of $\mathcal{C}'$, and outside of $\mathcal{C}$ is mapped to inside of $\mathcal{C}'$.



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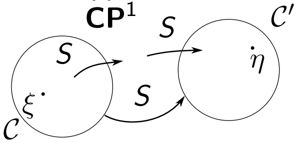

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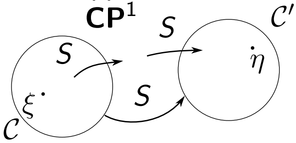

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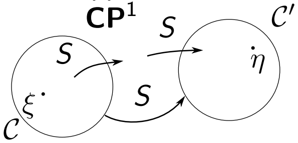

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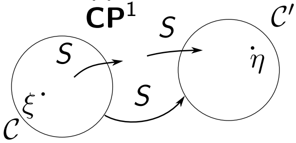

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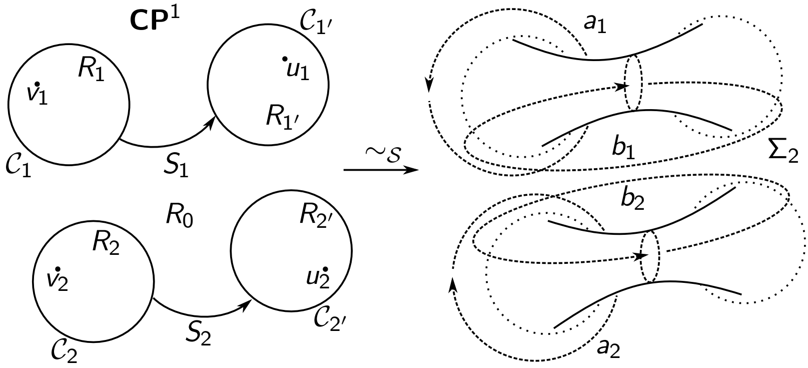
- *I.e.* we impose $z \sim T_\alpha(z)$ for **all** T_α in the *Schottky group*, the group of Möbius maps freely generated by $S_1, \ldots, S_g$.
- Nice geometric realization $(k_\mu, \eta_\mu, \xi_\mu)$ of $\dim(\mathcal{M}_g) = 3g - 3$.

The Schottky group



- The g-loop bosonic string measure (vacuum diagram) is given by

[Di Vecchia, Frau, Lerda, Sciuto 1987 Phys.Lett.B199]

$$Z_g = \int \frac{1}{\mathrm{d}V_{abc}} \prod_{\mu=1}^{g} \left(\frac{\mathrm{d}k_\mu \, \mathrm{d}\xi_\mu \, \mathrm{d}\eta_\mu}{k_\mu^2 (\eta_\mu - \xi_\mu)^2} \right) \frac{1}{(\det \operatorname{Im} \tau)^{D/2}}$$

$$\times \left(\prod_{\alpha}{}' \prod_{n=1}^{\infty} (1 - k_\alpha^n)^{-D+2} \right) \left(\frac{\prod_{\mu=1}^{M} (1 - k_\mu)^2}{\prod_\alpha{}' (1 - k_\alpha)^2} \right) . \quad (1)$$

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- Expressed in terms of the fixed points of the g Schottky generators as well as the multipliers k_α of all Schottky group elements.

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- *E.g.*: the 'super-Riemann sphere' $\mathbf{CP}^{1|1}$ defined by quotienting $\mathbf{C}^{2|1} - \mathbf{0}$ by the equivalence $(w, z|\theta) \sim (\lambda w, \lambda z|\lambda\theta)$; $\lambda \in \mathbf{C}_*$.

- There is a superconformal generalization of Möbius maps, easiest to write down as $(2|1) \times (2|1)$ $\mathrm{OSp}(1|2)$ matrices acting on the homogenous coordinates of $\mathbf{CP}^{1|1}$:

$$\begin{pmatrix} z_1 \\ z_2 \\ \hline \theta \end{pmatrix} \mapsto \left(\begin{array}{cc|c} a & b & \alpha \\ c & d & \beta \\ \hline \gamma & \delta & e \end{array}\right) \begin{pmatrix} z_1 \\ z_2 \\ \hline \theta \end{pmatrix} \tag{2}$$

where (for superconformality)

$$\begin{pmatrix} -\delta \\ \gamma \end{pmatrix} = \sqrt{ad - bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3}$$

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- Independent of overall factor, so fix super-determinant to 1, yielding $(5|4) - (2|2) = 3|2$ parameters.

- As in the bosonic case, we can characterize a super-Möbius map by two super-fixed-points and one multiplier:

$$\frac{Z \doteq \mathbf{S}(U)}{Z \doteq \mathbf{S}(V)} = k \frac{Z \doteq U}{Z \doteq V} \tag{5}$$

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- ($z|\psi \mapsto Z \dot{-} U|\psi - \theta$ is superconformal).
- Geometric realization of the $2 \times (1|1) + (1|0) = 3|2$ parameters.

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- Quotienting by a super-Möbius map is equivalent to sewing two Neveu-Schwarz punctures at the two fixed points, *i.e.* finding SC coordinates systems $x|\theta$, $y|\psi$ which vanish at the punctures and setting [Witten 2012 arXiv:1209.5461]

$$xy = -k\,; \quad y\theta = k^{\frac{1}{2}}\psi\,; \quad x\psi = -k^{\frac{1}{2}}\theta\,; \quad \theta\psi = 0\,. \tag{7}$$

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- Super-fixed-points & multipliers minus $\mathrm{OSp}(1|2)$ gauge fixing gives geometric realization of $\dim(\mathfrak{M}_g) = 3g-3|2g-2$.

- The genus-g superstring vacuum amplitude for the NS sector is given by [Di Vecchia, Frau, Hornfeck, Lerda, Sciuto 1987 Phys.Lett.B211]

$$\widehat{Z}_g = \int \frac{1}{\mathrm{d}V_{abc}} \prod_{\mu=1}^{g} \left(\frac{\mathrm{d}k_\mu}{k_\mu^{3/2}} \frac{\mathrm{d}V_\mu \, \mathrm{d}U_\mu}{V_\mu \doteq U_\mu} \frac{(1-k_\mu)^2}{(1-k_\mu^{1/2})^2} \right) \frac{1}{(\det \mathrm{Im}\, \boldsymbol{\tau})^{D/2}}$$
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- The integral includes a $2g-2$-dimensional Berezin integral.

- The genus-g superstring vacuum amplitude for the NS sector is given by [Di Vecchia, Frau, Hornfeck, Lerda, Sciuto 1987 Phys.Lett.B211]

$$\widehat{Z}_g = \int \frac{1}{\mathrm{d}V_{abc}} \prod_{\mu=1}^{g} \left(\frac{\mathrm{d}k_\mu}{k_\mu^{3/2}} \frac{\mathrm{d}V_\mu \, \mathrm{d}U_\mu}{V_\mu \doteq U_\mu} \frac{(1-k_\mu)^2}{(1-k_\mu^{1/2})^2} \right) \frac{1}{(\det \operatorname{Im} \boldsymbol{\tau})^{D/2}}$$
$$\times \prod_{\alpha}{}' \left[\prod_{n=1}^{\infty} \left(\frac{1-k_\alpha^{n-1/2}}{1-k_\alpha^n} \right)^D \prod_{n=2}^{\infty} \left(\frac{1-k_\alpha^n}{1-k_\alpha^{n-1/2}} \right)^2 \right] . \quad (8)$$

- Depends on fixed points of generators, period matrix $\boldsymbol{\tau}$, and multipliers of all SSG elements.
- The integral includes a $2g-2$-dimensional Berezin integral.
- Signs of $k_\mu^{1/2}$ are to be summed; this implements the Gliozzi-Scherk-Olive projection (in the NS sector).

Presentation Outline

The string theory setup

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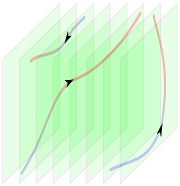

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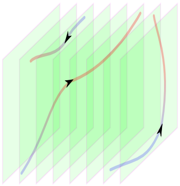

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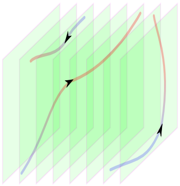

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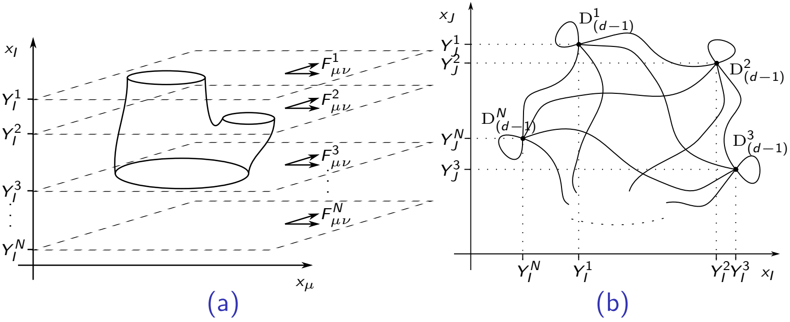
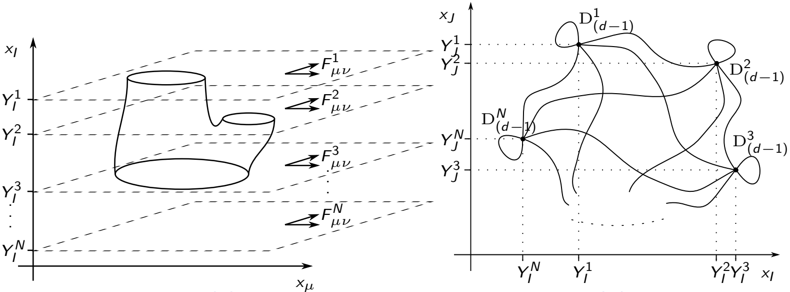
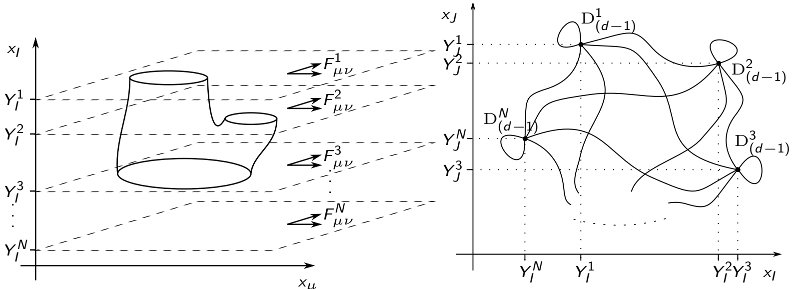
- We look only at the NS sector of open strings, so we only get the bosonic sector of $\mathcal{N}=4$, *i.e.* gauge fields (from parallel string modes) & 6 adjoint scalars (perpendicular modes).
- Separating branes breaks $U(N) \to U(1)^N$ and gives masses.


x_I
Y_I^1
Y_I^2
Y_I^3
Y_I^N
$F_{\mu\nu}^1$
$F_{\mu\nu}^2$
$F_{\mu\nu}^3$
$F_{\mu\nu}^N$
x_μ
(a)
x_J
Y_J^1
Y_J^2
Y_J^N
Y_J^3
$D^1_{(d-1)}$
$D^2_{(d-1)}$
$D^3_{(d-1)}$
$D^N_{(d-1)}$
Y_I^N
Y_I^1
Y_I^2
Y_I^3
x_I
(b)



(c) (d)

- We put *commuting* U(1) constant background fields on the brane world volumes.



(e) (f)

- We put *commuting* U(1) constant background fields on the brane world volumes.
- The D-branes are separated arbitrarily, giving masses $m_{ij}^2 = |(\vec{Y}_i - \vec{Y}_j)|^2/(2\pi\alpha')^2$ to strings between i'th, j'th branes.

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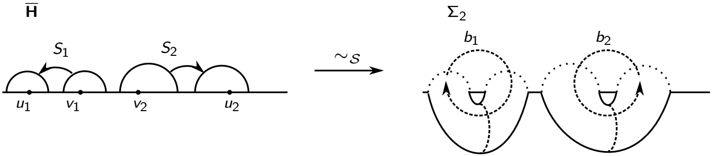

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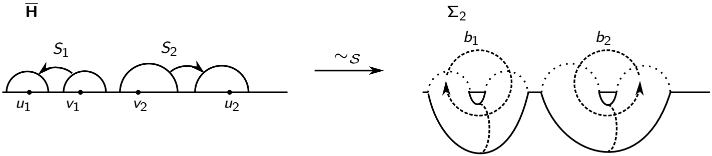

- The supermoduli space $\mathfrak{M}_2$ now has *real* dimension $3g - 3|2g - 2 = 3|2$.

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- The effect of separating the D-branes is to insert a factor

$$\mathcal{Y} = \prod_{i=1}^{N_\mathrm{s}} \mathrm{e}^{2\pi \mathrm{i}\, \alpha' \vec{m}_I \cdot \boldsymbol{\tau} \cdot \vec{m}_I}$$

where $\vec{m}_I \equiv (m_I^{13}, m_I^{23})$ with $m_I^{ab} \equiv (Y_I^a - Y_I^b)/(2\pi\alpha')$ encodes the distances between the 3 D-branes to which the worldsheet is attached. [Frau, Lerda, Pesando, Russo, Sciuto 1997]

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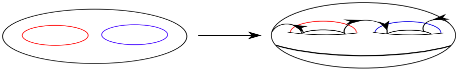

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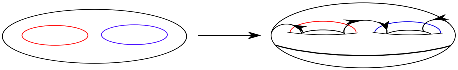
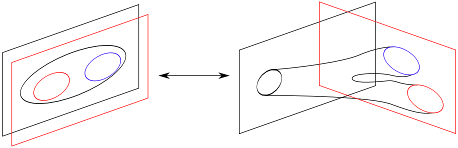


- Hard to implement the background fields directly, but can use T-duality with closed strings propagating between D-branes at an angle ϵ & sew explicitly. [Russo, Sciuto 2003]



- The background fields can be implemented in the string amplitude by including the ϵ-dependent factor [Russo, Sciuto 2004 & 2007; Magnea, Russo, Sciuto 2004]:

$$\mathcal{R}(\vec{\epsilon}) = \mathrm{e}^{-\mathrm{i}\pi\vec{\epsilon}\cdot\boldsymbol{\tau}\cdot\vec{\epsilon}} \frac{\det\left(\mathrm{Im}\,\boldsymbol{\tau}\right)}{\det\left(\mathrm{Im}\,\boldsymbol{\tau}_{\vec{\epsilon}}\right)} \prod_{\alpha}{}' \prod_{n=1}^{\infty} \left\{ \left(\frac{1-k_\alpha^{n-\frac{1}{2}}}{1-k_\alpha^{n}} \right)^{-2}, \right.$$

$$\left. \times \; \frac{\left(1-\mathrm{e}^{2\mathrm{i}\pi\vec{\epsilon}\cdot\boldsymbol{\tau}\cdot\vec{N}_\alpha}\, k_\alpha^{n-\frac{1}{2}}\right)\left(1-\mathrm{e}^{-2\mathrm{i}\pi\,\vec{\epsilon}\cdot\boldsymbol{\tau}\cdot\vec{N}_\alpha}\, k_\alpha^{n-\frac{1}{2}}\right)}{\left(1-\mathrm{e}^{2\pi\mathrm{i}\vec{\epsilon}\cdot\boldsymbol{\tau}\cdot\vec{N}_\alpha}\, k_\alpha^{n}\right)\left(1-\mathrm{e}^{-2\pi\mathrm{i}\vec{\epsilon}\cdot\boldsymbol{\tau}\cdot\vec{N}_\alpha}\, k_\alpha^{n}\right)} \right\}.$$

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- The 'twisted determinant' $\det(\mathrm{Im}\,\boldsymbol{\tau}_{\vec{\epsilon}})$ is an ϵ-dependent generalization of $\det(\mathrm{Im}\,\boldsymbol{\tau})$.

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$$\mathcal{Z}_2 = \prod_{\alpha}{}' \left[\prod_{n=1}^{\infty} \left(\frac{1-k_\alpha^{n-1/2}}{1-k_\alpha^n} \right)^2 \right]$$

then

$$\mathcal{Z}_{\parallel} = \frac{\mathcal{R}(\vec{\epsilon})\mathcal{Z}_2}{(\det \mathrm{Im}\,\boldsymbol{\tau})}\,; \quad \mathcal{Z}_{\perp} = \left(\frac{\mathcal{Z}_2}{\det \mathrm{Im}\,\boldsymbol{\tau}} \right)^{\frac{d-2}{2}}\,; \quad \mathcal{Z}_{\rm sc} = \mathcal{Y}(\vec{m}_I)(\mathcal{Z}_2)^{\frac{10-d}{2}}$$

and $\mathcal{Z}_{\rm bc}$ = everything else.

Presentation Outline

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- In the Dirichlet directions $M = I$ we put $\partial_I \mapsto 0$ and give the background field a VEV $A_I \mapsto \frac{1}{g} M_I$ and write $Q_I = \Phi_I$, a scalar.
- In the Neumann directions (parallel to the D brane) we put a background field $A_\mu(x) = B x_1 \eta_{\mu 2}$ which gives a constant field strength $F_{\mu\nu} = B(\eta_{\mu 1}\eta_{\nu 2} - \eta_{\mu 2}\eta_{\nu 1})$.

- It has been known since the early days of string theory that the divergences of string theory give Yang-Mills theory in the "Gervais-Neveu" gauge $\partial_M Q^M + i\gamma g Q_M Q^M = 0$.[Gervais, Neveu 1972]

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- Secondly, we impose the gauge condition *before* dimensional reduction, so afterwards it looks like

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- This gives us *e.g.* $mc\overline{c}\Phi$ vertices needed for matching with string theory.

- In our background $A_\mu(x) = Bx_1\eta_{\mu 2}$, the scalar and gluon *position space* propagators can be written down exactly in B in terms of a heat kernel [Magnea, Russo, Sciuto 2004; Ritus 1977]:

$$G^{ij}(x,y) = \int_0^\infty \mathrm{d}t\, \mathcal{K}^{ij}(x,y;t)$$

$$G^{ij}_{\mu\nu}(x,y) = -\int_0^\infty \mathrm{d}t\, \exp(2igF^{ij}t)_{\mu\nu}\, \mathcal{K}^{ij}(x,y;t)$$

$$\mathcal{K}^{ij}(x,y;t) = \frac{\mathrm{e}^{-\frac{i}{2}gB^{ij}(x_1+y_1)(x_2-y_2)-tm_{ij}^2}}{(4\pi t)^{\frac{d}{2}}}\frac{gB^{ij}t}{\sinh(gB^{ij}t)}$$
$$\times \exp\left[\frac{1}{4}(x_\mu - y_\mu)\beta(F^{ij}, t)^{\mu\nu}(x_\nu - y_\nu)\right]$$

$$\beta(F^{ij}, t)^{\mu\nu} = \mathrm{diag}\left(\frac{1}{t}, \frac{gB^{ij}}{\tanh(gB^{ij}t)}, \frac{gB^{ij}}{\tanh(gB^{ij}t)}, \frac{1}{t}, \ldots, \frac{1}{t}\right).$$

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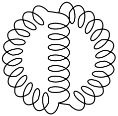

$$= \frac{g^2}{(4\pi)^d}\int_0^\infty \frac{\prod_{i=1}^3 \mathrm{d}t_i\, \mathrm{e}^{-t_i m_i^2}}{\Delta_0^{d/2-1}\Delta_F}\Bigg\{ -\frac{3-\gamma^2}{2}\frac{(d-2)(d-3)}{\Delta_0}(t_1+t_2+t_3)$$

$$-\frac{2(d-2)}{\Delta_F}\left(\frac{\sinh(gF_1 t_1)}{gF_1}\left(\frac{1-\gamma^2}{2}\cosh(gB_2t_2-gB_3t_3)\right.\right.$$

$$\left.\left.+\cosh(2gB_1t_1-gB_2t_2-gB_3t_3)+\text{cyclic permutations}\right)\right)$$

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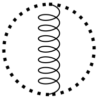
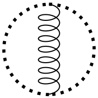


$$= \frac{1+\gamma^2}{2}\frac{g^2}{(4\pi)^d}\int_0^\infty \frac{\prod_{i=1}^3 \mathrm{d}t_i\, \mathrm{e}^{-t_i m_i^2}}{\Delta_0^{d/2-1}\Delta_F}\left\{\frac{2}{\Delta_F}\frac{\sinh(F_3 t_3)}{F_3}\cosh(2F_3 t_3 - F_1 t_1 - F_2 t_2) + \frac{d-2}{\Delta_0} t_3 + \text{ cyclic permutations }\right\}$$



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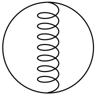
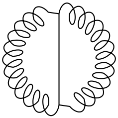
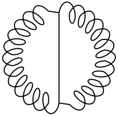


$$= -N_\mathrm{s} \frac{g^2}{(4\pi)^d} \int_0^\infty \frac{\prod_{i=1}^3 \mathrm{d}t_i \, \mathrm{e}^{-t_i m_i^2}}{\Delta_0^{d/2-1} \Delta_F} \left\{ \frac{d-2}{\Delta_0} \left(t_3 + \frac{1-\gamma^2}{4}(t_1+t_2) \right) + \frac{2}{\Delta_F} \left(\frac{\sinh(gB_3 t_3)}{gB_3} \cosh(2gB_3 t_3 - gB_1 t_1 - gB_2 t_2) + \frac{1-\gamma^2}{4} \left(\frac{\sinh(gB_1 t_1)}{gB_1} \cosh(gB_3 t_3 - gB_2 t_2) + \frac{\sinh(gB_2 t_2)}{gB_2} \cosh(gB_3 t_3 - gB_1 t_1) \right) \right) + \text{ cyclic permutations} \right\}$$



$$= i \, \frac{g^2}{(4\pi)^d} \int_0^\infty \frac{\prod_{i=1}^3 \mathrm{d}t_i \, \mathrm{e}^{-t_i m_i^2}}{\Delta_0^{d/2-1} \Delta_F} \Big\{ \Big(\frac{1+\gamma^2}{2} m_3^2 - m_1^2 - m_2^2 \Big) \times \big(d - 2 + 2\cosh(2gB_1 t_1 - 2gB_2 t_2) \big) + \text{ cyclic permutations} \Big\} \, .$$



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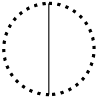

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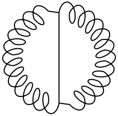

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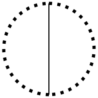

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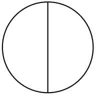
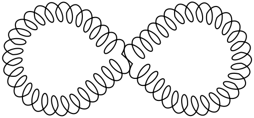


$$= i\,(N_{\mathrm{s}}-1)\,\frac{3-\gamma^2}{2}(m_1^2+m_2^2+m_3^2)\,\frac{g^2}{(4\pi)^d}\int_0^\infty \frac{\prod_{i=1}^3 \mathrm{d}t_i\,\mathrm{e}^{-t_i m_i^2}}{\Delta_0^{d/2-1}\Delta_F}\,.$$



$$= i\,\frac{g^2}{(4\pi)^d}\int_0^\infty\Big[\prod_{i=1}^{2}\frac{\mathrm{d}t_i\,\mathrm{e}^{-t_i m_i^2}\,gB_i}{t_i^{d/2-1}\,\sinh(gB_i t_i)}\Big]$$

$$\times\,\frac{1}{2}\Big\{d-2+2\cosh(2gB_1t_1+2gB_2t_2)$$

$$+\,\frac{\gamma^2-1}{2}\Big(d-2+2\cosh(2gB_1t_1-2gB_2t_2)$$

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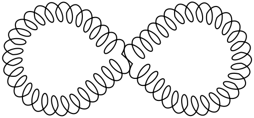

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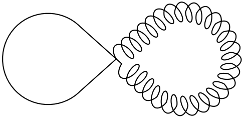

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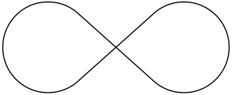

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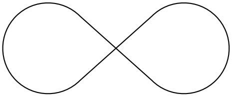

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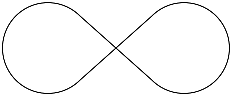

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- Some *e.g.* the scalar-gluon Fig. of 8 completely vanish.

Presentation Outline

The field theory limit

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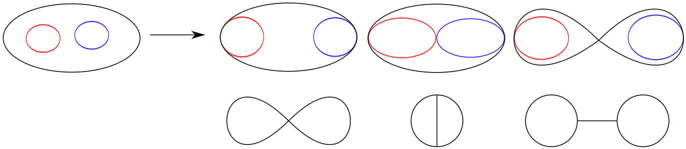
- The $\alpha' \to 0$ field theory limit arises from the boundary of moduli space where string worldsheets degenerate.



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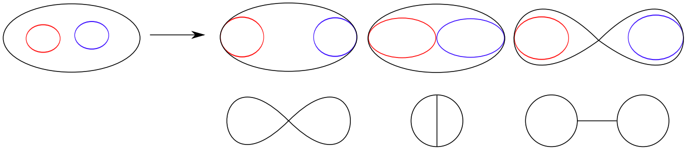

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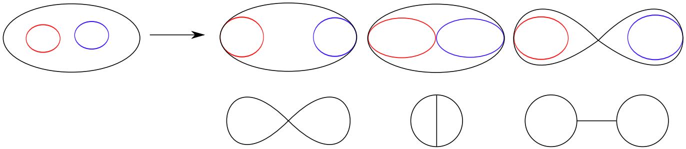

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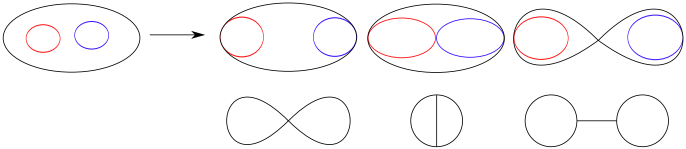

- Degeneration points $\leftrightarrow$ edges in the corresponding Feynman graph.
- Heuristically, multipliers $k_\mu \leftrightarrow$ Schwinger parameters t_μ via $k_\mu = \mathrm{e}^{-\frac{t_\mu}{\alpha'}}$.
- Only finitely many terms survive the $\alpha' \to 0$ limit: should get the field theory amplitude exactly.

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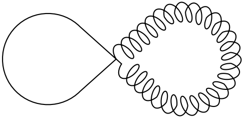
- The factor we take $k_\mu^{\frac{1}{2}}$ from determines the field propagating in the corresponding leg of the Feynman diagram.

- For example, to get the Feynman diagram



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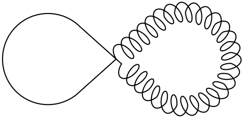
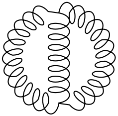


  we take $k_1^{1/2}$ from $\mathcal{Z}_{sc}$ and $k_2^{1/2}$ from $\mathcal{Z}_{\perp}$ or $\mathcal{Z}_{\parallel}$.
- The coefficients of $k_\mu^{1/2}$ in those terms determine the structure of the Feynman graph.

- Some of the Feynman graphs we want to obtain have a topology with a 3-fold symmetry *e.g.*



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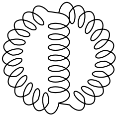
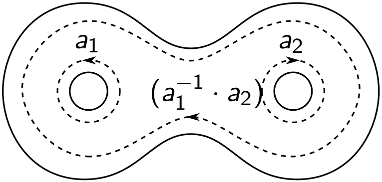


- The 3 bosonic worldsheet moduli we are integrating over are not symmetric (2 multipliers from the two handles; 1 anharmonic ratio of the fixed points) are not symmetric, so how can we map them onto Schwinger times?

- The solution is to recognize that the worldsheet has a 3-fold symmetry between homology cycles:



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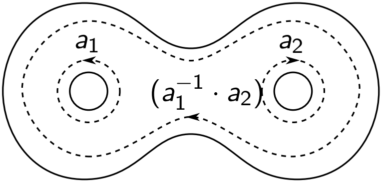

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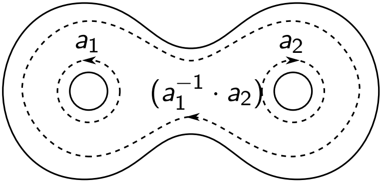

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- So choose as the bosonic moduli p_i, $i = 1, 2, 3$ where

$$p_1 p_3 = k_1 \, ; \qquad p_2 p_3 = k_2 \, ; \qquad p_1 p_2 = k(\mathbf{S}_1^{-1}\mathbf{S}_2) \, ,$$

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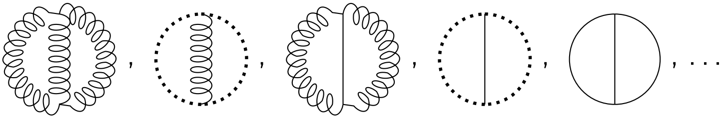
- Unlike with bosonic strings, the $\mathrm{d}p_i/p_i^{3/2}$ pole disappears after Grassmann integration + GSO projection (so no tachyons).

- After carrying out this procedure mechanically, we obtain exactly all the Feynman graphs with this topology, in GN gauge $\gamma^2 = 1$:



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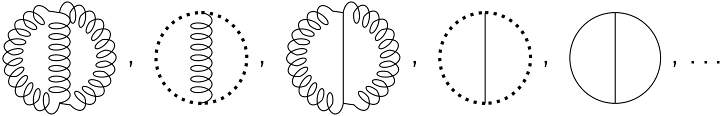

- Diagrams with an odd number of scalars originate from sub-leading terms in the expansion of

$$\mathrm{e}^{2\pi i \vec{m}_I \cdot \boldsymbol{\tau} \cdot \vec{m}_I} = p_1^{\alpha' m_1^2} p_2^{\alpha' m_2^2} p_3^{\alpha' m_3^2} \Big(1 + \alpha' m_3^2 \theta_1 \theta_2 \big(\sqrt{p_3} + \sqrt{p_1 p_2 p_3}\big) + \ldots\Big)$$

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- *E.g.* the integral $\int \mathrm{d}u\, \mathrm{d}\theta\, \mathrm{d}\phi\, \frac{1}{u} = \int \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}(\log u)$ differs by 1 depending on whether we take the lower limit as $u = 0$ or $p_3 \sim u(1 + \theta\phi) = 0$ (despite the same leading behaviour).

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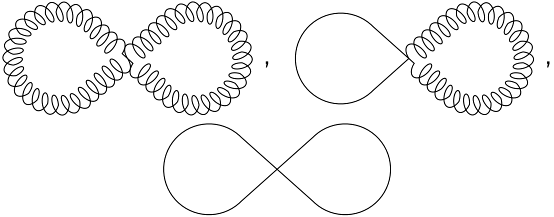

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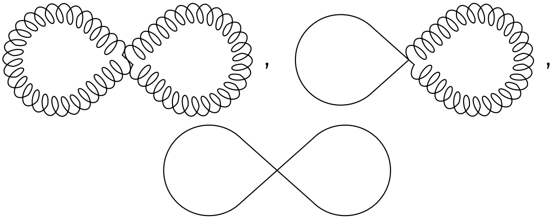
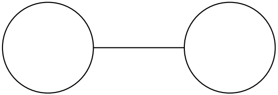


- the $1/(1 - u + \theta\phi)$ term gives a contribution which has the structure of 1PR diagrams but with a wrong factor of 2—maybe the wrong integration limit (*cf.* u *vs.* p_3).



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- This should allow us to compute *e.g.* the effective action for Yang-Mills quite directly from string theory.
- Direction in which the work can be extended: spacetime fermions (Ramond sector), gravity amplitudes (with closed strings), more loops.

Grazie!