Thesis submitted for the degree of Doctor of Philosophy

Gauge theory effective actions from open strings

Author:
Samuel Rhys Playle

Supervisor:
Dr. Rodolfo Russo

September 2014

Abstract

In an introductory chapter, a summary of the construction of string theories is given, for both the bosonic string and the RNS superstring. Relevant mathematical technology is introduced, including super-Riemann surfaces. Conformal field theory is discussed and BRST quantization of the string is explained.

(Super) Schottky groups for the construction of higher-genus Riemann surfaces are introduced. As an example of the use of Schottky groups and super-Riemann surfaces, the one-loop gluon two point function is calculated from string theory.

The incorporation of background gauge fields into string theory *via* nontrivial monodromies (twists) is discussed. The two loop Prym period matrix determinant is computed in the Schottky parametrization.

The string theory model with N parallel separated D_3-branes is introduced, and the formulae for the the vacuum amplitude are written down. A manifestly symmetric parametrization of two loop Schottky space is introduced. The relationship between worldsheet moduli and Feynman graph Schwinger times is given. The $\alpha' \to 0$ limit of the amplitude is written down explicitly.

The lagrangian for the corresponding gauge theory is found, making use of a generalization of Gervais-Neveu gauge which accounts for scalar VEVs. Propagators in the given gauge field background are written down. All of the 1PI two-loop Feynman diagrams are written down, including diagrams with vertices with an odd number of scalars. Illustrative example Feynman graphs are computed explicitly in position space. These results are compared with the preceding string theory results and exact agreement is obtained for the 1PI diagrams.

An example application is given: the computation of the β function of scalar QED at two loops with the same methods, leading to the same result as found in the literature.

Acknowledgements

My Ph.D. studies, including the research in this thesis, have been funded by a Science and Technology Facilities Council (STFC) Postgraduate Studentship.

My research has benefitted from helpful comments and suggestions of several people. Firstly, I would like to thank my supervisor Dr. Rodolfo Russo whose guidance and teaching has been crucial to my studies and the completion of the work herein. Engaging discussions with my other collaborators, Professor Lorenzo Magnea and Professor Stefano Sciuto, have also been vital to the development of my ideas.

Thanks are also due to my academic colleagues at Queen Mary University of London, in particular my fellow Ph.D. students who have been a source of lively and inspirational discussion, in particular Joel Fitzhardinge-Berkeley, David Garner, Ömer Gürdoğan, Edward Hughes, Zac Kenton, Dimitrios Korres, Paolo Mattioli, Moritz McGarrie, James McGrane, Robert Mooney, Edvard Musaev, Jurgis Pasukonis, Brenda Penante, and Felix Rudolph.

Lastly I wish to thank my friends and my family for their patient support throughout my studies, in particular my parents Kathryn and Tom to whom this thesis is dedicated.

Declaration

I, Samuel Rhys Playle, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with or supported by others, this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

I attest that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge break any UK law, infringe any third party's copyright or other intellectual property right, or contain any confidential material.

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Details of collaboration and publications:

This thesis describes research carried out in collaboration with Lorenzo Magnea, Rodolfo Russo, and Stefano Sciuto. Some of the results of this work have been published in [1] and a forthcoming article will contain the results in Chapters 4 and 5. Where other sources of information have been used, they have been cited in the bibliography.

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Chapter 1

Introduction

As has been known since the early days of string theory [2, 3], in the limit of vanishing string length (or infinite tension) $\alpha' \to 0$, the theory can be approximated by a low-energy effective theory of (super) Yang-Mills coupled to Einstein gravity.

In this thesis, we establish that the correspondence holds not only at the level of the full amplitude, but even diagram-by-diagram, with the contributions associated with the various quantum fields being identifiable on the string theory side. To make the correspondence, we need to use a particular parametrization of (super) moduli space known as Schottky groups, which have been in use since the very early days of string theory [4, 5, 6] and set on a more rigourous footing in the 1980s [7, 8, 9, 10]. On the quantum field theory (QFT) side, we need to use a particular non-linear gauge condition first written down by Gervais and Neveu in [11].

String theories have been a source of useful insights about gauge theories [12, 13, 14, 15] and we discuss, in particular, the calculation of the Euler-Heisenberg effective action for scalar QED using string-based techniques as in [16].

The structure of the thesis is as follows: in Chapter 2 we recall some aspects of the construction of string theories, starting with bosonic strings and then superstrings in the Ramond-Neveu-Schwarz formalism. We discuss various aspects of quantization, such as the construction of the worldsheet Faddeev-Popov (b, c) ghost system and BRST quantization. We show how (super) Riemann surfaces are constructed with the use of (super) Schottky groups, and as an example of the use of super Schottky groups, we give some details of the calculation of the one-loop correction to the gluon two-point function from string theory.

In chapter 3, we discuss how a constant background U(1) gauge field can be incorporated into string theory by giving non-trivial monodromies to the worldsheet fields, and we calculate the determinant of the super Prym period matrix, which is an important ingredient for the two loop amplitudes we calculate in the following chapter.

In chapter 4, we discuss the model we use as a 'laboratory' to examine the correspondence, namely a stack of N parallel separated D_3-branes with constant U(1) gauge fields on their worldvolume. We show how to calculate the two-loop vacuum amplitude in this context. We then discuss how to find the $\alpha' \to 0$ limit in way that makes the diagram-by-diagram matching with QFT completely manifest.

In Chapter 5, we find the lagrangian for the Yang-Mills theory which corresponds to the low-energy effective theory of our setup, and then we calculate all of the two-loop 1PI vacuum diagrams. We show how all of them match terms in the Schottky group expansion of the string theory amplitude in the previous Chapter.

In Chapter 6, we discuss how the correspondence can be used as a tool for investigation QFT effective actions, and as an example we discuss how the two loop Callan-Symanzik β function for scalar QED would be studied with our approach. We finish with Chapter 7 which discusses forthcoming work and possible generalizations of the research we've undertaken.

Chapter 2

Multiloop calculations in Superstring theory

In this chapter we make extensive use of the textbooks [17, 18, 19] and the lecture notes [20, 21].

An account of the historical development of, and motivation for string theory is given in [22].

String theory evolved originally from a 4-point scattering amplitude written down by Veneziano [23] constructed to satisfy the properties of crossing symmetry and of being expressible as a sum of poles in either the s-channel or the t-channel (but not both channels simultaneously) [24]. The expression was quickly generalized to an arbitrary number of external states (see the early reviews [25, 26] and references therein). It was realized soon by a number of authors that the scattering amplitudes described the dynamics of relativistic strings ([27] and references therein).

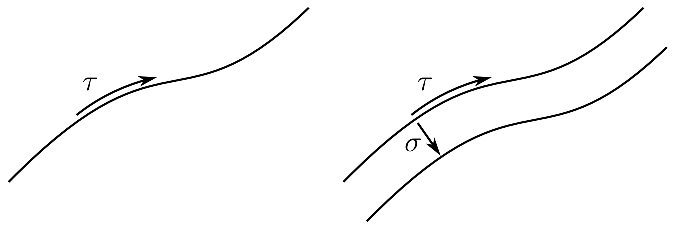
2.1 Classical bosonic strings and superstrings

2.1.1 Classical point particles

To guide our analysis of quantum strings, we begin with the much simpler but in some ways analogous case of the point particle. Classically, the Lorentz-invariant action for a point-like scalar particle of mass m is simply m times the relativistically-invariant length of the worldline:

$$S = -m \int \mathrm{d}\tau \, \sqrt{\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau}} \, , \tag{2.1}$$

where τ is a co-ordinate on the worldline of the particle (see Fig. 2.1a), and $X^\mu(\tau)$ is the embedding function giving the particle's position in spacetime. Eq. (2.1) has the property of being invariant under reparametrizations of the worldline $\tau \to \tau'$, but since it is proportional to m it vanishes identically for massless particles. This can be remedied



(a) (b)

Figure 2.1: Co-ordinates on the worldline of a particle and the worldsheet of a string.

by introducing another worldline field $e(\tau)$; then introducing the action

$$S = -\frac{1}{2}\int d\tau \left(\frac{1}{e(\tau)}\frac{\partial X^\mu}{\partial \tau}\frac{\partial X^\nu}{\partial \tau}\eta_{\mu\nu} + m^2\, e(\tau)\right). \tag{2.2}$$

To find the classical equations of motion (e.o.m.), this must be extremized with respect to the variations $e \to e + \delta e$ and $X^\mu \to X^\mu + \delta X^\mu$, yielding

$$\frac{\partial X^\mu}{\partial \tau}\frac{\partial X^\nu}{\partial \tau}\eta_{\mu\nu} - e^2 m^2 = 0 \tag{2.3}$$

and

$$\frac{\partial}{\partial \tau}\left(\frac{1}{e}\frac{\partial X^\nu}{\partial \tau}\right) = 0\,, \tag{2.4}$$

respectively. Now, Eq. (2.2) is still invariant under an arbitrary reparametrization $\tau \to \tilde{\tau}$, so long as $e(\tau)$ transforms as $e(\tilde{\tau}) = \left(\frac{\partial \tilde{\tau}}{\partial \tau}\right)^{-1} e(\tau)$. We can use this redundancy to choose τ such that $e(\tau) \equiv 1$, then the e.o.m. Eq. (2.4) take on the same form as those for the original action Eq. (2.1) with τ as the proper time. Because of this we say that the actions are 'classically equivalent'.

2.1.2 Classical bosonic strings

The first attempt at writing down the action for a free bosonic string involved the natural generalization of Eq. (2.1), namely, it was proportional to the Lorentz-invariant "area" of the two-dimensional surface traced out by the one-dimensional string as it propagated through spacetime, multiplied by the tension of the string [28]. We can parametrize the 1-dimensional string by a coordinate σ and let the two-dimensional surface traced out by the string, *i.e.* the *worldsheet* Σ, be parametrized by σ and another coordinate τ as in Fig. 2.1b. Then the action is given in terms of the pullback (or 'induced') metric by the *Nambu-Goto action*: if we let $\sqrt{-a}$ denote $\sqrt{-\det a}$ for any bilinear form a, then

$$S_{\text{Nambu-Goto}} = -T\int d\tau\, d\sigma\, \sqrt{-\gamma}\,; \qquad \gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta}\eta_{\mu\nu} \tag{2.5}$$

where $T \equiv \frac{1}{2\pi\alpha'}$ is the string tension, and $\sigma^0 \equiv \tau$; $\sigma^1 \equiv \sigma$. This is difficult to quantize because of the square root, so as we did in the point particle case, we introduce a new field $h^{\alpha\beta}(\sigma,\tau)$ and define the action [29]

$$S_{\text{bos}} = -\frac{T}{2}\int \mathrm{d}\sigma\,\mathrm{d}\tau\,\sqrt{-h}\,h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\,. \tag{2.6}$$

To see that Eq. (2.6) is classically equivalent to Eq. (2.5), we can impose the principle of stationary action with respect to $h^{\alpha\beta}$. The variation of S_{bos} can be found with the aid of the Jacobi rule for the variation of a determinant, $\delta\sqrt{-h} = -\frac{1}{2}\sqrt{-h}h_{\alpha\beta}\delta h^{\alpha\beta}$, yielding

$$\delta S_{\text{bos}} = -\frac{T}{2}\int \mathrm{d}\sigma\,\mathrm{d}\tau\,\sqrt{-h}\,\Big(-\frac{1}{2}h_{\gamma\delta}h^{\alpha\beta} + \delta^\alpha_\gamma\delta^\beta_\delta\Big)\delta h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\,. \tag{2.7}$$

Customarily, we write [17]

$$T^X_{\alpha\beta} \equiv -\frac{4\pi\alpha'}{\sqrt{-h}}\frac{\delta S_{\text{bos}}}{\delta h^{\alpha\beta}} \tag{2.8}$$

$$= \Big(-\frac{1}{2}h_{\alpha\beta}h^{\gamma\delta} + \delta^\gamma_\alpha\delta^\delta_\beta\Big)\partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu}\,, \tag{2.9}$$

so the vanishing of the variation of the action with respect to $h^{\alpha\beta}$ can be imposed by the constraint $T^X_{\alpha\beta} = 0$. $T^X_{\alpha\beta}$ is called the stress-energy tensor for X^μ, since its definition matches that in general relativity. $T^X_{\alpha\beta}$ vanishes, *i.e.* the equations of motion are satisfied, whenever $h^{\alpha\beta}$ is proportional to the pullback metric up to rescaling:

$$h^{\alpha\beta}(\sigma,\tau) = \Omega(\sigma,\tau)\,\partial^\alpha X^\mu \partial^\beta X^\nu \eta_{\mu\nu} \qquad \Rightarrow \qquad \frac{\delta S_{\text{bos}}}{\delta h^{\alpha\beta}} = 0\,. \tag{2.10}$$

Inserting this expression for $h^{\alpha\beta}$ into S_{bos}, we retrieve the Nambu-Goto action Eq. (2.5) as expected.

In fact, the rescaling in Eq. (2.10) is not just a symmetry of the classical solutions but a symmetry of the action Eq. (2.6) called Weyl invariance [29], under which $h_{\alpha\beta}$ can be locally rescaled:

$$h^{\alpha\beta}(\sigma,\tau) \to \mathrm{e}^{\widetilde{\phi}(\sigma,\tau)}h^{\alpha\beta}(\sigma,\tau)\,. \tag{2.11}$$

$T^X_{\alpha\beta}$ is automatically traceless due to Weyl invariance. To see this, we consider an infinitesimal Weyl symmetry $h^{\alpha\beta} \to (1-\delta\lambda)h^{\alpha\beta}$, then [30]

$$T^\alpha_{X\alpha} = T^X_{\alpha\beta}h^{\alpha\beta} = -\frac{4\pi}{\sqrt{-h}}\frac{\delta S_{\text{bos}}}{\delta h^{\alpha\beta}}\frac{\delta h^{\alpha\beta}}{\delta\lambda} = -\frac{4\pi}{\sqrt{-h}}\frac{\delta S_{\text{bos}}}{\delta\lambda} = 0 \tag{2.12}$$

where we've used the fact that the Weyl transformation affects S_{bos} only via $h^{\alpha\beta}$.

It can be readily seen that the action Eq. (2.6) is also invariant under reparametrization of the worldsheet. To derive constraints, we need to consider infinitesimal diffeomorphisms. Consider a vector field $\epsilon\xi^\alpha$ which deforms the co-ordinates as $\sigma^\alpha \to \sigma^\alpha + \epsilon\xi^\alpha$, then to first order in ϵ, the variation of tensors on the worldsheet is given by the Lie derivative $\mathcal{L}_\xi$ with respect to ξ [31]. The fields transform as [17, 32, 31]

$$X^\mu \to X^\mu + \epsilon\mathcal{L}_\xi(X^\mu) \qquad = X^\mu + \epsilon\xi^\alpha\partial_\alpha X^\mu\,;$$

$$h_{\alpha\beta} \to h_{\alpha\beta} + \epsilon \mathcal{L}_\xi(h_{\alpha\beta}) \qquad = h_{\alpha\beta} + \epsilon \nabla_\alpha \xi_\beta + \epsilon \nabla_\beta \xi_\alpha \,, \tag{2.13}$$

to order $\mathcal{O}(\epsilon^2)$. ∇_α is the Levi-Civita connection which is symmetric and satisfies $\nabla_\xi h_{\alpha\beta} = 0$, given in terms of Christoffel symbols by [32]

$$\nabla_\alpha \xi_\beta = \partial_\alpha \xi_\beta - \Gamma^\gamma{}_{\alpha\beta}\, \xi_\gamma \,; \qquad \Gamma^\gamma{}_{\alpha\beta} = \frac{1}{2} h^{\gamma\delta}(\partial_\alpha h_{\beta\delta} + \partial_\beta h_{\alpha\delta} - \partial_\delta h_{\alpha\beta}) \tag{2.14}$$

To find the equation of motion, we can make a convenient choice for $h^{\alpha\beta}$ using Eq. (2.13) and Eq. (2.11) to put it in the form

$$h_{\alpha\beta}(\sigma,\tau) = \mathrm{e}^{\phi(\sigma,\tau)} \eta_{\alpha\beta} \,. \tag{2.15}$$

It is always possible to choose coordinates for which $\mathrm{e}^\phi \equiv 1$ locally, but there are topological obstructions for more complicated worldsheets. In the gauge Eq. (2.15), the action becomes

$$S_{\text{conf}} = -\frac{T}{2} \int \mathrm{d}\sigma\, \mathrm{d}\tau\, \partial^\alpha X^\mu \partial_\alpha X^\nu\, \eta_{\mu\nu} \,, \tag{2.16}$$

from which it's easy to see that the e.o.m. for X^μ is the wave equation $\partial^\alpha \partial_\alpha X^\mu = 0$. More generally, the e.o.m. can be written in terms of the Laplacian Δ as [33]

$$\Delta X^\nu = 0 \,; \qquad \Delta \equiv -\frac{1}{\sqrt{-h}} \partial_\alpha \Big(\sqrt{-h}\, h^{\alpha\beta} \partial_\beta\, \cdot \Big) \,. \tag{2.17}$$

Note that the action is independent of the overall factor e^ϕ appearing in Eq. (2.15), corresponding to the Weyl symmetry. It follows from diffeomorphism invariance that $T^{\alpha\beta}$ is covariantly conserved whenever the e.o.m. for X^μ are satisfied. To see this, we consider the variation of Eq. (2.6) under a reparametrization generated by the vector field ξ^α, so X^μ and $h_{\alpha\beta}$ vary according to Eq. (2.13). Since we know that this is a symmetry of S, we have

$$0 = \delta S = \frac{\delta S}{\delta X^\mu} \delta X^\mu + \frac{\delta S}{\delta h_{\alpha\beta}} \delta h_{\alpha\beta} \tag{2.18}$$

$$= -\frac{1}{2\pi\alpha'} \int \mathrm{d}\tau \int \mathrm{d}\sigma\, \sqrt{-h}\, T_X^{\alpha\beta}\, \nabla_\alpha \xi_\beta \,, \tag{2.19}$$

where we've used $\frac{\delta S}{\delta X^\mu} = 0$ and inserted the expression for $T_{\alpha\beta}$ in Eq. (2.8) and the expression for $\delta h_{\alpha\beta}$ in Eq. (2.13). Partially integrating the covariant derivative on the right-hand side of Eq. (2.19) we get

$$\nabla_\alpha T_X^{\alpha\beta} = 0 \,, \tag{2.20}$$

where the covariant derivative of a tensor with two upper indices is given by

$$\nabla_\lambda T^{\alpha\beta} = \partial_\lambda T^{\alpha\beta} + \Gamma^\alpha{}_{\gamma\lambda} T^{\gamma\beta} + \Gamma^\beta{}_{\gamma\lambda} T^{\alpha\gamma} \,. \tag{2.21}$$

The action Eq. (2.6) enjoys a *conformal symmetry* following from Weyl invariance

Eq. (2.11) and reparametrization invariance. A conformal transformation is a diffeomorphism $\sigma^\alpha \mapsto \sigma'^\alpha = f^\alpha(\sigma)$ such that the metric is unchanged only up to the action of a Weyl transformation, $\mathrm{d}s'^2 = \Omega(\sigma)\mathrm{d}s^2$. In terms of an infinitesimal conformal transformation with $f^\alpha(\sigma) = \sigma^\alpha + \epsilon\xi^\alpha$ and $\Omega(\sigma) = 1 + \epsilon\kappa$, to first order in ϵ we have

$$\kappa h_{\alpha\beta} = h_{\lambda\beta}\,\partial_\alpha\xi^\lambda + h_{\alpha\lambda}\,\partial_\beta\xi^\lambda + \xi^\lambda\partial_\lambda h_{\alpha\beta}\,, \tag{2.22}$$

which is the conformal Killing equation [34]. The right-hand side can be written in terms of the connection as $\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha$ or in terms of the Lie derivative as $\mathcal{L}_\xi(h_{\alpha\beta})$. For $\kappa = 0$ it is just the Killing equation for isometries. Taking the trace of both sides by multiplying with $h^{\alpha\beta}$, we get $2\kappa = 2\nabla_\alpha\xi^\alpha$, which we can substitute back into Eq. (2.22) obtaining

$$P_1(\xi)_{\alpha\beta} \equiv \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha - h_{\alpha\beta}\nabla_\lambda\xi^\lambda = 0\,. \tag{2.23}$$

Here we've defined the operator P_1 which maps vectors fields to traceless symmetric tensors; its kernel it the space of conformal killing vectors [33, 35, 36].

In conformal coordinates we have:

$$P_1(\xi) = h_{\lambda\beta}\,\partial_\alpha\xi^\lambda + h_{\alpha\lambda}\,\partial_\beta\xi^\lambda - h_{\alpha\beta}\,\partial_\lambda\xi^\lambda = 0\,. \tag{2.24}$$

Vector fields satisfying this equation generate symmetries of the string action in conformal gauge, Eq. (2.16). Multiplying both sides of Eq. (2.24) by $\partial^\alpha = h^{\alpha\gamma}\partial_\gamma$, we see that the terms proportional to $\partial_\gamma\phi$ all cancel since they are just proportional to Eq. (2.24), two of the remaining terms cancel and we are left with

$$\partial^\alpha\partial_\alpha\xi_\beta = 0\,. \tag{2.25}$$

We also need to introduce boundary conditions in the σ direction. In the case of a closed string, topologically a circle, we customarily let the domain of σ be $[0, 2\pi]$ so we typically impose the boundary condition $X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau)$ to ensure that X^μ is single valued (although this is not necessarily true, for example, when the target space is an orbifold, X^μ is only single-valued modulo the action of a discrete group [37]).

Alternatively, the string can have the topology of a line interval; in this case we let σ range over $\sigma \in [0, \pi]$. In this case, when we vary X^μ we pick up a boundary contribution:

$$\delta S_{\mathrm{bos}} = -T\int_{-\infty}^{\infty}\mathrm{d}\tau\,\Big[\partial^\sigma X^\mu\delta X^\nu\Big]_{\sigma=0}^{\sigma=\pi}\eta_{\mu\nu} + T\int_{-\infty}^{\infty}\mathrm{d}\tau\int_0^\pi \mathrm{d}\sigma\,\delta X^\mu\partial_\alpha\partial^\alpha X^\nu\eta_{\mu\nu}\,, \tag{2.26}$$

where we've assumed that $\delta X^\mu(\sigma, \pm\infty) \to 0$. Therefore for the action to be stationary, we need not only the equation of motion $\partial_\alpha\partial^\alpha X^\mu = 0$, but also conditions to be met at the boundary. We can choose either

$$\delta X^\mu\Big|_{\sigma=0,\pi} = 0 \qquad\qquad \text{or} \qquad\qquad \partial^\sigma X^\nu\Big|_{\sigma=0,\pi} = 0\,; \tag{2.27}$$

the first choice, in which X^μ is fixed at the endpoint, is called a *Dirichlet* boundary condition; the second choice is called a *Neumann* boundary condition. The two endpoints can have their boundary conditions chosen independently.

We will typically work with a Euclideanized worldsheet action by Wick rotating $\tau \to -\mathrm{i}\,\tau$ so the worldsheet metric has signature $(+,+)$. The metric Eq. (2.15) becomes $h^{\alpha\beta} = \mathrm{e}^{\phi}\delta^{\alpha\beta}$ and it is useful to introduce the complex coordinates

$$z = \frac{\tau + \mathrm{i}\,\sigma}{\sqrt{2}}\,; \qquad \overline{z} = \frac{\tau - \mathrm{i}\,\sigma}{\sqrt{2}}\,, \tag{2.28}$$

so $\overline{z} = z^*$. The derivatives transform as

$$\partial_\tau = \frac{1}{\sqrt{2}}(\partial + \overline{\partial})\,; \qquad \partial_\sigma = \frac{\mathrm{i}}{\sqrt{2}}(\partial - \overline{\partial})\,, \qquad \text{where} \qquad \partial \equiv \frac{\partial}{\partial z}\,; \qquad \overline{\partial} \equiv \frac{\partial}{\partial \overline{z}}\,. \tag{2.29}$$

The diagonal components of the metric in this basis vanish: we have

$$h_{zz} = \langle \partial, \partial \rangle = \frac{1}{2}\mathrm{e}^{\phi}\langle \partial_\tau - \mathrm{i}\partial_\sigma, \partial_\tau - \mathrm{i}\partial_\sigma \rangle = 0 \tag{2.30}$$

and similarly $h_{\overline{z}\overline{z}} = 0$, while

$$h_{z\overline{z}} = \langle \partial, \overline{\partial} \rangle = \frac{1}{2}\mathrm{e}^{\phi}\langle \partial_\tau - \mathrm{i}\partial_\sigma, \partial_\tau + \mathrm{i}\partial_\sigma \rangle = \mathrm{e}^{\phi} = h_{\overline{z}z}\,, \tag{2.31}$$

so the metric has the form

$$h_{z\overline{z}} = h_{\overline{z}z} = \mathrm{e}^{\phi(z,\overline{z})}\,; \qquad h_{zz} = h_{\overline{z}\overline{z}} = 0\,. \tag{2.32}$$

The trace of $T^X_{\alpha\beta}$ in these coordinates is therefore equal to

$$T^X_{\alpha\beta}h^{\alpha\beta} = 2\mathrm{e}^{-\phi}T^X_{\overline{z}z} = 0\,, \tag{2.33}$$

i.e. $T^X_{\alpha\beta}$ has been diagonalized. The Christoffel symbols for Eq. (2.32) can be calculated from Eq. (2.14); all vanish except

$$\Gamma^z{}_{zz} = \partial\phi\,; \qquad \Gamma^{\overline{z}}{}_{\overline{z}\overline{z}} = \overline{\partial}\phi\,. \tag{2.34}$$

The z component of the conservation equation Eq. (2.20) then becomes

$$\begin{aligned} 0 = \nabla_\alpha T^{\alpha z}_X &= \partial T^{zz}_X + 2\partial\phi T^{zz}_X \\ &= \mathrm{e}^{-2\phi}\partial(\mathrm{e}^{2\phi}T^{zz}_X) = \mathrm{e}^{-2\phi}\partial(h_{z\overline{z}}h_{z\overline{z}}T^{zz}_X) = \mathrm{e}^{-2\phi}\partial(h_{\overline{z}\alpha}h_{\overline{z}\beta}T^{\alpha\beta}_X) = \mathrm{e}^{-2\phi}\partial\, T^X_{\overline{z}\overline{z}}\,, \end{aligned} \tag{2.35}$$

so $\partial T^X_{\overline{z}\overline{z}} = 0$, *i.e.* $T^X_{\overline{z}\overline{z}}$ is anti-holomorphic (at least away from other operator insertions [20]). Similarly, from the $\overline{z}$ component of Eq. (2.20) we see that $\overline{\partial} T^X_{zz} = 0$, so T^X_{zz} is holomorphic. Using the fact that this implies $T^X_{zz}(z,\overline{z})$ has only trivial dependence on $\overline{z}$, and similarly that $T^X_{\overline{z}\overline{z}}(z,\overline{z})$ has only trivial dependence on z, we write

$$T^X(z) \equiv T^X_{zz}(z,\overline{z})\,; \qquad \overline{T}^X(\overline{z}) \equiv T^X_{\overline{z}\overline{z}}(z,\overline{z})\,. \tag{2.36}$$

In these coordinates, the string action has the form

$$S_{\rm bos} = T \int {\rm d}z\, {\rm d}\overline{z}\, \partial X^\mu\, \overline{\partial} X^\nu\, \eta_{\mu\nu}\,. \tag{2.37}$$

where a factor of i from the Wick rotation has been absorbed to keep the action positive-definite. The action is unchanged by a coordinate transformation of the form

$$z \mapsto z' = f(z,\overline{z})\,; \qquad \overline{z} \mapsto \overline{z}' = f(z,\overline{z})^*\,, \qquad \text{where} \qquad \overline{\partial} f(z,\overline{z}) = 0\,. \tag{2.38}$$

i.e. a holomorphic change of coordinates. From this point onwards, we will write a holomorphic function as $f(z)$, as though we are treating z and $\overline{z}$ as separate variables. Eq. (2.37) has the e.o.m. $\partial\overline{\partial}X^\mu = 0$ whose general solution is a sum of arbitrary 'left-moving' holomorphic and 'right-moving' anti-holomorphic parts

$$X^\mu(z,\overline{z}) = X^\mu_{\rm L}(z) + X^\mu_{\rm R}(\overline{z})\,. \tag{2.39}$$

In these coordinates with the metric Eq. (2.32), the conformal killing vectors can be found from Eq. (2.24); from $\alpha = \beta = z$ we get $\partial\xi^{\overline{z}} = 0$ so $\xi^{\overline{z}}$ is anti-holomorphic and from $\alpha = \beta = \overline{z}$ we get $\overline{\partial}\xi^z = 0$ so ξ^z is holomorphic, while from $\alpha = z$, $\beta = \overline{z}$ we get $\partial\xi^z = \overline{\partial}\xi^{\overline{z}}$.

The Noether currents associated to conformal transformations can be expressed in terms of the stress-energy tensor $T^X(z)$, $\overline{T}^X(\overline{z})$. To see this, we begin by considering an infinitesimal diffeomorphism $\sigma^\alpha \mapsto \sigma^\alpha + \epsilon\,\xi^\alpha(\sigma)$, not in general a reparametrization because it's not accompanied by a corresponding change in the metric $\delta h_{\alpha\beta} = \epsilon(\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha)$. We know that the action Eq. (2.37) is unchanged by a reparametrization, which means that the action will change under an infinitesimal diffeomorphism by *minus* the change in the action coming from the change in the metric which would be required to make the infinitesimal diffeomorphism a reparametrization. Therefore, the change in the action coming from the infinitesimal diffeomorphism $\sigma^\alpha \mapsto \sigma^\alpha + \epsilon\,\xi^\alpha(\sigma)$ will be equal to the change in the action coming from making *no* change to the coordinates but deforming the metric by $h_{\alpha\beta} \mapsto h_{\alpha\beta} - \epsilon(\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha)$. We can write the change in the action coming from a change in the metric in terms of the stress-energy tensor, thanks to Eq. (2.8). The deformation of the inverse metric can be written in terms of the deformation of the metric via $\delta h^{\alpha\beta} = -h^{\alpha\gamma}h^{\beta\delta}\delta h_{\alpha\beta} = \epsilon(h^{\alpha\gamma}\nabla_\gamma\,\xi^\beta + h^{\beta\delta}\nabla_\delta\,\xi^\alpha)$. The deformation of the action is given, then, by

$$\delta S = \frac{1}{2\pi\alpha'} \int {\rm d}^2\sigma\, \sqrt{h}\, T^X_{\alpha\beta}\, \epsilon\, h^{\alpha\gamma}\nabla_\gamma\, \xi^\beta\,, \tag{2.40}$$

where we've used the symmetry of $T^X_{\alpha\beta}$. In terms of the complex coordinates in Eq. (2.28) with the conformal-gauge metric Eq. (2.32), this can be written as

$$\delta S = \frac{1}{2\pi\alpha'} \int {\rm d}^2 z\, \epsilon(T^X\overline{\partial}\xi^z + \overline{T}^X\partial\xi^{\overline{z}})\,, \tag{2.41}$$

where the scaling factors from the inverse metric $h^{\alpha\gamma}$ and from $\sqrt{h}$ have cancelled out, we've used $T^X_{z\overline{z}} = 0$, and we've used that $\nabla_z \xi^{\overline{z}} = \partial \xi^{\overline{z}}$ and $\nabla_{\overline{z}} \xi^z = \overline{\partial} \xi^z$ since the Christoffel symbols vanish (Eq. (2.34)).

Treating z and $\overline{z}$ as independent coordinates, we may vary them independently, so we may set *e.g.* $\xi^z = f(z, \overline{z})$ and $\xi^{\overline{z}} = 0$. In this case we get from Eq. (2.41) that

$$\delta S = \epsilon \int \mathrm{d}^2 z \, J^\alpha \partial_\alpha f(z, \overline{z}) \, ; \quad \text{where} \quad J^z(z, \overline{z}) = 0 \quad \text{and} \quad J^{\overline{z}}(z, \overline{z}) = \frac{1}{2\pi} T^X(z) \, , \tag{2.42}$$

i.e. $T(z)$ is the conserved current associated to translations in z. But more generally, we can find infinitely many conserved currents: if we multiply $f(z, \overline{z})$ by an arbitrary holomorphic function $h(z)$, then it passes through $\overline{\partial}$ in Eq. (2.41) and we get the conserved current

$$J^z_h(z, \overline{z}) = 0 \, ; \qquad\qquad J^{\overline{z}}_h(z, \overline{z}) = \frac{1}{2\pi} T^X(z) h(z) \, , \tag{2.43}$$

which is holomorphic. Similarly, by setting $\xi^z = 0$ and $\xi^{\overline{z}} = \tilde{h}(\overline{z}) f(z, \overline{z})$ we get an anti-holomorphic conserved current $\overline{J}_{\tilde{h}}(z, \overline{z})$ with $\overline{J}^z_{\tilde{h}}(z, \overline{z}) = \frac{1}{2\pi\alpha'} \overline{T}^X(\overline{z}) \tilde{h}(\overline{z})$ and $\overline{J}^{\overline{z}}_{\tilde{h}}(z, \overline{z}) = 0$.

The two components of $T^X_{\alpha\beta}$ for the X^μ fields given by Eq. (2.9) can be written down in complex coordinates as

$$T^X(z) = -\frac{1}{\alpha'} \partial X^\mu \partial X^\nu \eta_{\mu\nu} \, ; \qquad\qquad \overline{T}^X(\overline{z}) = -\frac{1}{\alpha'} \overline{\partial} X^\mu \overline{\partial} X^\nu \eta_{\mu\nu} \, . \tag{2.44}$$

The coordinates Eq. (2.28) range over $\mathrm{Re}(z) \in \mathbf{R}$ and $\mathrm{Im}(z) \in [0, \frac{\pi}{\sqrt{2}}]$ for open strings and $[0, \sqrt{2}\pi]$ for closed strings. It is conventional to change coordinates to

$$z \mapsto z' = \mathrm{e}^{\sqrt{2} z} \, ; \qquad\qquad \overline{z} \mapsto \overline{z}' = \mathrm{e}^{\sqrt{2}\overline{z}} \, , \tag{2.45}$$

which is a holomorphic change of coordinates so the action Eq. (2.37) is unchanged. For open strings, the boundary $\sigma = 0$ is mapped to the positive real axis $z' = \mathrm{Re}(z') > 0$ and the boundary $\sigma = \pi$ is mapped to the negative real axis $z' = \mathrm{Re}(z') < 0$. The interior of the string is mapped onto the upper-half plane $\mathbf{H} = \{x + \mathrm{i}\, y | (x, y) \in \mathbf{R}^2; y > 0\}$. Closed strings, for which $\sigma \in [0, 2\pi)$, are mapped onto the whole complex plane. τ is mapped onto the radial coordinate $|z'| = \mathrm{e}^\tau$; the far future and past $\tau \to \pm\infty$ are mapped onto $z' = \infty$ and $z' = 0$. Constant τ slices of the worldsheet are mapped onto circles with constant $|z'|$.

2.1.3 Riemann surfaces

In fact, let us relax our original assumption that the worldsheet has the topology of a strip, and assume only that it is a surface, possibly with boundaries corresponding to the endpoints of open strings. *A priori* it could be a non-orientable surface (although we will see that the Type II superstring theories we are interested in contain only orientable worldsheets). Any orientable two-dimensional Riemannian manifold Σ (*i.e.* a surface with a Riemannian metric) has an *almost complex structure*, *i.e.* a tensor $J_m{}^n$ satisfying $J_m{}^n J_n{}^p = -\delta^p_m$; it is given by $J_m{}^n = \sqrt{h} \varepsilon_{mp} h^{pn}$ where ε_{mp} is the antisymmetric

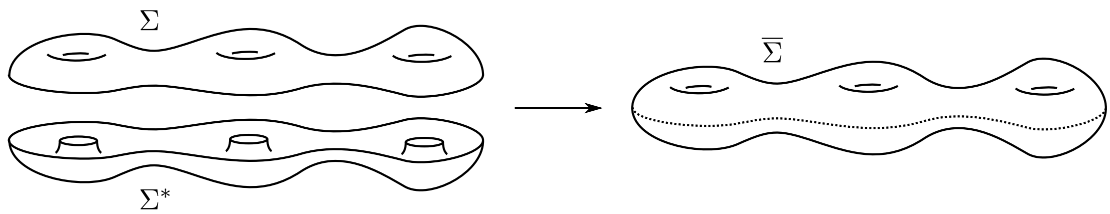


Figure 2.2: The double of a bordered Riemann surface.

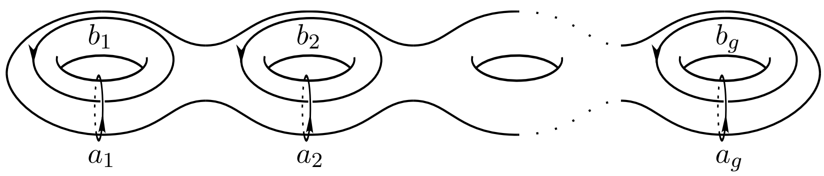


Figure 2.3: A basis for the first homology group on a compact Riemann surface.

symbol with $\varepsilon_{12} = 1$ [38, 39]; since the surface is two dimensional this is always integrable to a complex structure [38, 40].

In the case that the surface has no boundary, it is therefore a *Riemann surface*, *i.e.* a manifold of real dimension 2 which can be covered by *complex* coordinate charts whose transition functions are holomorphic [41, 42]. A function on a Riemann surface is holomorphic if it is holomorphic in a coordinate chart. Two Riemann surfaces are identified if there is a biholomorphic mapping between them. Since holomorphic transformations do not in general preserve a metric, there is no intrinsic metric on a Riemann surface. Riemann surfaces are a natural setting in which to consider the string action Eq. (2.37); the expression is unchanged by holomorphic changes of coordinates and therefore the integrand is globally well-defined.

Two dimensional surfaces with a complex structure and a boundary are called *bordered* Riemann surfaces by mathematicians [43, 44]. They are locally biholomorphic to the upper half plane $\overline{\mathbf{H}} \equiv \{z = x + \mathrm{i}\, y \in \mathbf{C} | y \geq 0\}$. To every bordered Riemann surface Σ we can associate a *double surface* $\overline{\Sigma}$ [6, 38]; this is defined by taking two copies of the surface and replacing each chart on the second copy with its complex conjugate, which maps points into the *lower* half plane, and then identifying corresponding points on the borders of the two bordered Riemann surfaces (see Fig. 2.2). The map $I : \Sigma \to \Sigma^*$ taking each point to its copy lifts to an anti-conformal involution on $\overline{\Sigma}$ whose fixed point set is the border.

To help us describe a compact Riemann surface Σ of genus g, we introduce a basis for its first homology group $H_1(\Sigma, \mathbf{Z}) \cong \mathbf{Z}^{2g}$ (which is the abelianization of its fundamental group). This is a set of $2g$ equivalence classes of curves a_μ and b_μ for $\mu = 1, \ldots, g$ such that no two a_i cycles and no two b_i cycles intersect each other, and that a_i intersects b_j once if and only if $i = j$ (see Fig. 2.3. We can make this precise with the introduction of an anti-symmetric *intersection form* [35] $(\cdot, \cdot) : H_1(\Sigma, \mathbf{Z}) \times H_1(\Sigma, \mathbf{Z}) \to \mathbf{Z}$ which counts the number of times the two curves intersect, with an opposite sign for oppositely oriented intersections.

There is a notion of a *complex line bundle* over a Riemann surface, that is, a 2-dimensional complex manifold E with a holomorphic projection map $\pi : E \to \Sigma$ such that π^{-1} of every point on Σ is a copy of $\mathbf{C}$ [45]. For an open neighbourhood U on Σ there is a local trivialization, *i.e.* $\pi^{-1}(U)$ is holomorphically equivalent to $U \times \mathbf{C}$. Roughly, E is a family of one-dimensional vector spaces varying holomorphically on Σ. Under pointwise tensor product of vector spaces, the holomorphic line bundles on Σ form a group called the *Picard group*; the inverse of a line bundle is the dual bundle and the identity is the trivial line bundle, *i.e.* the ring of meromorphic functions on Σ.

Line bundles can also be described in terms of equivalence classes of divisors: for a section[1] s of a line bundle $\xi \to \Sigma$ with n_ith-order poles at finitely many points P_i and m_jth-order zeroes at finitely-many points Q_j, the *divisor* of s is the formal sum $\mathrm{div}(s) = \sum_i n_i P_i - \sum_j m_j Q_j$, and its *degree* is given by $\deg(\mathrm{div}(s)) = \sum_i n_i - \sum_j n_j$. Multiplying the section s by an arbitrary meromorphic function, we can get any other section s' of ξ which may have different poles and zeroes from s with different multiplicities, however, because meromorphic functions have the same number of poles and zeroes, $\deg(\mathrm{div}(s')) = \deg(\mathrm{div}(s))$. Two divisors on Σ are in the same *divisor class* if they have the same degree, and there is a one-to-one correspondence between divisor classes and line bundles [35]. The space of holomorphic line bundles of degree D on Σ is called the *Picard variety* $\mathrm{Pic}_D(\Sigma)$.

The cotangent bundle of Σ can be decomposed as $T^*\Sigma = T^{*(1,0)}\Sigma \oplus T^{*(0,1)}\Sigma$ where $T^{*(1,0)}\Sigma$ and $T^{*(0,1)}\Sigma$ are the line bundles whose local trivializations are spanned by $\mathrm{d}z$ and $\mathrm{d}\bar{z}$, respectively [46]; these are well-defined because the transition functions on Σ are holomorphic. $T^{*(1,0)}\Sigma$ is a holomorphic line bundle called the *canonical bundle* and written K. There is a g-dimensional vector space of sections of K which are locally of the form $\omega_\mu = f_\mu(z)\mathrm{d}z$ where f_μ is holomorphic, called *abelian differentials of the first kind* [46].

We can decompose any tensor bundle on Σ into a direct sum of line bundles. Given a tensor C on Σ, pick local coordinates and a metric of the form Eq. (2.32), then each component of C will have a certain number of upper and lower z and $\bar{z}$ indices. We can use the metric to lower or raise upper or lower $\bar{z}$ indices to lower or upper z indices, respectively [47]. Then if a component $C^{z\cdots z}{}_{z\cdots z}$ has n_+ upper z indices and n_- lower z indices then it transforms under a map $z \mapsto z'$ as [48, 47]

$$C^{z\cdots z}{}_{z\cdots z} \to C^{z'\cdots z'}{}_{z'\cdots z'} = \left(\frac{\partial z'}{\partial z}\right)^{n_+ - n_-} C^{z\cdots z}{}_{z\cdots z} \,. \tag{2.46}$$

The set of objects which transform like Eq. (2.46) with $n_+ - n_- = n$ is a line bundle which is denoted K^n, and is equal to $K^{\otimes n}$ in the Picard group of Σ [35]. Although we used a metric to remove the $\bar{z}$ indices, the line bundle decomposition of the components only depends on the off-diagonal structure of the metric, which is the same for all metrics compatible with the complex structure.

Rotations in the (co)tangent spaces correspond to multiplication by complex phases: $\mathrm{d}z \to \mathrm{e}^{\mathrm{i}\theta}\mathrm{d}z$, $\partial \to \mathrm{e}^{-\mathrm{i}\theta}\partial$. The rank of a tensor is therefore characterized by its *helicity*: a section C of K^n transforms under rotation by θ as [47, 33]

$$C \to \exp(\mathrm{i}n\theta)C \,. \tag{2.47}$$

If C is a tensor in K^n then $h^{z\bar{z}}\partial_{\bar{z}}K$ is a tensor in K^{n+1}. Motivated by this, we can define

[1] recall that a section of a fibre bundle $\pi : E \to B$ is a map $s : B \to E$ such that $\pi \circ s = \mathrm{id}$.

covariant derivatives [47, 39] (with the sign for ∇^z_n opposite from [35])

$$\nabla^z_n : K^n \to K^{n+1}\,, \qquad\qquad \nabla_z^n : K^n \to K^{n-1}\,, \tag{2.48}$$

$$C \mapsto h^{z\bar{z}}\,\overline{\partial}\, C\,; \qquad\qquad C \mapsto (h^{z\bar{z}})^n \partial\big[(h_{\bar{z}z})^n C\big]\,. \tag{2.49}$$

The action of ∇_z^n on the helicity decomposition of a tensor is equivalent to the action of the Levi-Civita connection ∇_z on the tensor. Formally, the two operators in Eq. (2.48) are the adjoints of each other with respect to an inner product on K^n [47]:

$$(\nabla^z_n)^\dagger = -\nabla_z^{n+1}\,; \qquad\qquad \langle C_1, C_2\rangle_n = \int \mathrm{d}^2 z\,\sqrt{-h}(h_{\bar{z}z})^n\, C_1^* C_2\,. \tag{2.50}$$

The unique genus 0 Riemann surface is called the Riemann sphere; it can be constructed as the one-dimensional complex projective space $\mathbf{CP}^1$ obtained by quotienting the homogenous coordinates $(z_1, z_2) \in \mathbf{C}^2 \setminus \{0, 0\}$ by the equivalence relation $(z_1, z_2) \sim (\lambda z_1, \lambda z_2)$ for $\lambda \in \mathbf{C}^* \equiv \mathbf{C} \setminus \{0\} = \mathrm{GL}(1, \mathbf{C})$. The automorphisms of the Riemann sphere can be written as invertible linear transformations of the homogeneous coordinates:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\,; \qquad\qquad ad - bc \neq 0\,. \tag{2.51}$$

Since we can scale all of (a, b, c, d) by an overall factor and get the same map on $\mathbf{CP}^1$, we are free to choose *e.g.* $ad - bc = 1$; the group of these transformations is the projective special linear group $\mathrm{PSL}(2, \mathbf{C})$. $\mathbf{CP}^1$can be covered by two coordinate charts: for $z_2 \neq 0$ we can use $(z_1, z_2) \mapsto z = z_1/z_2 \in \mathbf{C}$ and for $z_1 \neq 0$ we can use $(z_1, z_2) \mapsto w = z_2/z_1 \in \mathbf{C}$. In terms of the co-ordinate z, Eq. (2.51) takes the form

$$z \mapsto z' = \frac{az + b}{cz + d}\,, \tag{2.52}$$

called a *Möbius transformation* or a *fractional linear transformation*.

Every compact Riemann surface Σ of genus g has a simply-connected universal covering surface $\widetilde{\Sigma}$ such that $\Sigma = \widetilde{\Sigma}/\Gamma$ where Γ is a group of Möbius transformations (this is called the 'uniformization theorem') [41]. $\widetilde{\Sigma}$ is either the Riemann sphere if $g = 0$, the complex plane $\mathbf{C}$ if $g = 1$, or the upper-half plane $\mathbf{H}$ if $g \geq 2$. For $g \geq 2$, the Möbius transformations in Γ have to leave the boundary of $\mathbf{H}$, *i.e.* the real line $\mathbf{R}$, fixed; therefore the entries in the Möbius map must be real; a group of Möbius maps all satisfying this property is called *Fuchsian* [49]. For $g \geq 1$, choosing a fundamental domain for the action of Γ (*i.e.* a connected subset of $\widetilde{\Sigma}$ containing one representative of each equivalence class of Γ) allows us to find a single coordinate z covering all of Σ.

One way to construct a fundamental domain is by fixing a point P on Σ and finding a set of $2g$ curves on Σ equivalent to the canonical homology basis which start and end at P, with no intersections anywhere except P (see Fig. 2.4a for $g = 2$), and then 'cut Σ open' along the curves, obtaining topologically a polygon with $4g$ edges (see Fig. 2.4a for $g = 2$). The cutting procedure for $g = 2$ is shown step-by-step in Fig. 12 of [39]. Any of the lifts of the polygon to $\widetilde{\Sigma}$ is a fundamental domain for Γ; in this case it is called a

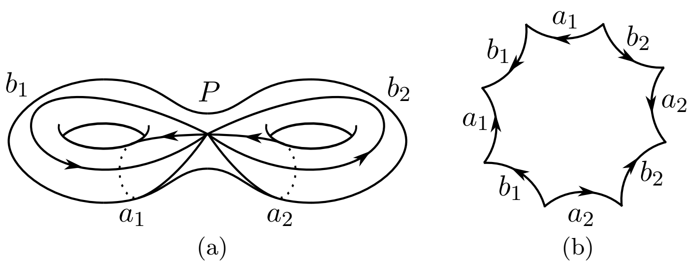


Figure 2.4: Cutting open a $g = 2$ surface to find a fundamental polygon.

fundamental polygon [50]. Γ is isomorphic to the fundamental group $\pi_1(\Sigma, P)$. Because of the existence of the fundamental polygon, the homotopy class $a_1^{-1}b_1^{-1}a_1b_1 \cdots a_g^{-1}b_g^{-1}a_gb_g$ is contractible, and therefore $\pi_1(\Sigma, P)$ has a non-trivial group relation.

Given any basis $\{\phi_i\}$ of abelian differentials with periods around the a_i cycles given by $(\tau_a)_{ij} = \int_{a_i} \phi_j$, we can find a basis $\{\omega_i\}$ given by $\omega_i = (\tau_a^{-1})_{ij}\phi_j$ which is dual to the a_i cycles; we can then use the integrals of ω_i around the b-cycles to define the *period matrix* τ_{ij}:

$$\int_{a_i} \omega_j = \delta_{ij}\,; \qquad\qquad \int_{b_i} \omega_j = \tau_{ij}\,. \tag{2.53}$$

τ_{ij} is symmetric and has positive-definite imaginary part. These statements can be proved by considering area integrals over the fundamental polygon and equating them by Stokes' theorem to integrals over the homology cycles; the first follows by integrating $\omega_i \wedge \omega_j$ which is a total derivative and the second follows by integrating $\eta \wedge \overline{\eta} = -2\,\mathrm{i}\,|h|^2 \mathrm{d}x \wedge \mathrm{d}y$ where $\eta(z) = h(z)\mathrm{d}z = \sum_{i=1}^{g} c_i \omega_i(z)$ is some abelian differential [50, 35].

Given a Riemann surface Σ with a choice of homology basis, we can define the *jacobian torus* in terms of the period matrix τ_{ij} [35]:

$$J(\Sigma) \equiv \mathbf{C}^g / L_\tau\,; \qquad L_\tau \equiv \{(\tau \cdot \vec{n}) + \vec{m}\,|\,\vec{n}, \vec{m} \in \mathbf{Z}^g\} = \mathbf{Z}^g + \tau\,\mathbf{Z}^g\,. \tag{2.54}$$

$J(\Sigma)$ has a natural complex structure. After choosing an arbitrary base point p_0 on Σ, we can define a map (the *Jacobi map*) [35]

$$\Phi^i_{p_0} : \Sigma \to J(\Sigma)\,; \qquad\qquad p \mapsto \int_{p_0}^{p} \omega_i\,. \tag{2.55}$$

The ϑ*-function* is a section of a holomorphic line bundle $E \to J(\Sigma)$ given by the formula

$$\vartheta(\vec{z};\tau) \equiv \sum_{\vec{n}\in\mathbf{Z}^g} \exp\left(\mathrm{i}\pi\vec{n}\cdot\tau\cdot\vec{n} + 2\pi\mathrm{i}\,\vec{n}\cdot\vec{z}\right). \tag{2.56}$$

The ϑ *function with characteristics* is given by

$$\vartheta\left[\begin{array}{c}\vec{a}\\ \vec{b}\end{array}\right](\vec{z};\tau) \equiv \sum_{\vec{n}\in\mathbf{Z}^g} \mathrm{e}^{\mathrm{i}\pi(\vec{n}+\vec{a})\cdot\tau\cdot(\vec{n}+\vec{a})+2\pi\mathrm{i}\,(\vec{n}+\vec{a})\cdot(\vec{z}+\vec{b})} \tag{2.57}$$

$$= \mathrm{e}^{\mathrm{i}\pi\vec{a}\cdot\tau\cdot\vec{a}+2\pi\mathrm{i}\,\vec{a}\cdot(\vec{z}+\vec{b})}\,\vartheta(\vec{z}+\tau\cdot\vec{a}+\vec{b}\,;\tau)\,. \tag{2.58}$$

Its behaviour under translations in L_τ of its argument or its characteristics is given by:

$$\vartheta\Big[\begin{array}{c}\vec{a}\\ \vec{b}\end{array}\Big](\vec{z}+\tau\cdot\vec{n}+\vec{m}\,;\tau) = \mathrm{e}^{-\mathrm{i}\pi\vec{n}\cdot\tau\cdot\vec{n}-2\pi\mathrm{i}(\vec{z}+\vec{b})+2\pi\mathrm{i}\vec{a}\cdot\vec{m}}\,\vartheta\Big[\begin{array}{c}\vec{a}\\ \vec{b}\end{array}\Big](\vec{z}\,;\tau)\,, \tag{2.59}$$

$$\vartheta\Big[\begin{array}{c}\vec{a}+\vec{m}\\ \vec{b}+\vec{n}\end{array}\Big](\vec{z}\,;\tau) = \mathrm{e}^{2\pi\mathrm{i}\vec{n}\cdot\vec{m}}\,\vartheta\Big[\begin{array}{c}\vec{a}\\ \vec{b}\end{array}\Big](\vec{z}\,;\tau)\,. \tag{2.60}$$

There is a function which is well-defined on the cut surface

$$f(p) = \vartheta(\vec{z}+\vec{\Phi}_{p_0}(p)\,;\tau)\,; \tag{2.61}$$

according to the *Riemann vanishing theorem* either $f(p) = 0$ for all $p \in \Sigma$ or f has g zeroes p_i satisfying

$$\vec{z} + \sum_{i=1}^{g}\vec{\Phi}_{p_0}(p_i) = \vec{\Delta}_{p_0} \tag{2.62}$$

where the *vector of Riemann constants* $\vec{\Delta}_{p_0} \in J(\Sigma)$ is independent of $\vec{z}$ [35].

From the Riemann vanishing theorem, it follows that any meromorphic function on Σ may be written in terms of its divisor $D = z_1 + \ldots + z_d - w_1 - \ldots - w_d$ as [20]:

$$f_D(p) = \prod_{i=1}^{d}\vartheta(\vec{\zeta}+\vec{\Phi}_{z_i}(p)\,;\tau)\,\vartheta(\vec{\zeta}+\vec{\Phi}_{w_i}(p)\,;\tau)^{-1}\,, \tag{2.63}$$

where $\vec{\zeta} \in J(\Sigma)$ is any point such that $\vartheta(\vec{\zeta}\,;\tau) = 0$; f_D is independent of $\vec{\zeta}$.

A *spin bundle* on a Riemann surface is a line bundle whose square is the canonical bundle K. A section s of the spin bundle lifts to a section $\widetilde{s}$ of the trivial bundle on the covering surface $\widetilde{\Sigma}$ which transforms under the action of the covering group Γ as $s\circ T = \pm s$ depending on the 'spin structure' associated to a homology cycle T [46, 35].

We can define the *prime form* [44] which is a holomorphic $(-1/2,0)\times(-1/2,0)$-differential on $\Sigma\times\Sigma$ given by [20]:

$$E(z,w) \equiv \frac{\vartheta[{\vec{a}\atop\vec{b}}]\big(\int_w^z\vec{\omega}\,;\tau\big)}{h_{[{\vec{a}\atop\vec{b}}]}(z)h_{[{\vec{a}\atop\vec{b}}]}(w)}\,, \qquad h_{[{\vec{a}\atop\vec{b}}]}(z) = \Big(\sum_{i=1}^{g}\frac{\partial}{\partial\zeta^i}\vartheta[{\vec{a}\atop\vec{b}}](\vec{\zeta}\,;\tau)\Big|_{\zeta^i=0}\omega_i(z)\Big)^{\frac{1}{2}}\,, \tag{2.64}$$

where $[{\vec{a}\atop\vec{b}}]$ is any *odd* half-characteristic, *i.e.* $a_i, b_i \in \{0,\frac{1}{2}\}$ and $4\vec{a}\cdot\vec{b}$ is odd. $E(z,w)$ is odd under swapping its arguments and its limiting behaviour is given by $E(z,w) \sim (z-w)(\mathrm{d}z)^{-\frac{1}{2}}(\mathrm{d}w)^{-\frac{1}{2}}$ as $z \to w$ [51]. $E(z,w)$ picks up a minus sign as z or w move around an a_i cycle; as z moves around the homology cycle b_i, $E(z,w)$ changes as [20]

$$E(z,w) \to \exp\big[-\pi\mathrm{i}\tau_{ii} - 2\pi\mathrm{i}\int_z^w\omega_i\big]\,E(z,w)\,. \tag{2.65}$$

2.1.4 The classical Ramond-Neveu-Schwarz superstring

The bosonic string action Eq. (2.6) can be extended to be supersymmetric on the worldsheet. We need to introduce spinors on the worldsheet, so we need a representation ρ^α of the Clifford algebra satisfying $\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}$, where we will use a Lorentzian metric with signature $(-+)$ at first. In string theory there is variation in the sign convention for the Clifford algebra; we follow [32, 17, 52] and differ from [18, 53, 54]. A suitable basis can be constructed from the Pauli matrices τ^i:

$$\rho^0 = \tau^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}; \qquad \rho^1 = \mathrm{i}\,\tau^1 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}; \tag{2.66}$$

they are chosen to be imaginary so that the Dirac operator will be real. The chirality matrix is diagonal $\rho^3 = \rho^0\rho^1 = \tau^3$, so the two components of a spinor have opposite chirality:

$$P_\pm = \frac{1 \mp \rho^3}{2}; \qquad P_-\chi = \begin{pmatrix} \chi_- \\ 0 \end{pmatrix} \qquad P_+\chi = \begin{pmatrix} 0 \\ \chi_+ \end{pmatrix}. \tag{2.67}$$

where the projection operators are idempotent $P_\pm^2 = P_\pm$. The sole generator of spin$(1,1)$ is $\rho^{01} \equiv \frac{1}{2}[\rho^0, \rho^1]$ is equal to the chirality operator $\rho^{01} = \rho^3$; vectors transform under boosts as

$$v^\alpha\rho_\alpha \mapsto v'^\alpha\rho_\alpha = \exp(\frac{\theta}{2}\rho^{01})v^\alpha\rho_\alpha\exp(-\frac{\theta}{2}\rho^{01}) \tag{2.68}$$

which is the same as setting

$$v^0 + v^1 \mapsto \mathrm{e}^\theta(v^0 + v^1); \qquad v^0 - v^1 \mapsto \mathrm{e}^{-\theta}(v^0 - v^1); \tag{2.69}$$

i.e. the vector representation is not irreducible but splits into two representations of opposite chirality. Spinors transform under the same boost by $\chi \to \chi' = \mathrm{e}^{\frac{\theta}{2}\rho^{01}}\chi$, so the two components transform as

$$\chi_+ \to \chi'_+ = \mathrm{e}^{\frac{\theta}{2}}\chi_+; \qquad \chi_- \to \chi'_- = \mathrm{e}^{-\frac{\theta}{2}}\chi_-. \tag{2.70}$$

The worldsheet action for a superstring can be obtained from the action for the bosonic string in conformal gauge Eq. (2.16) by adding a Dirac term for a worldsheet Majorana-Weyl spinor ψ; we set [55, 32]

$$S = -\frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma\,\mathrm{d}\tau\,\left(\partial_\alpha X^\mu\partial^\alpha X^\nu - \mathrm{i}\,\overline{\psi}^\mu\rho^\alpha\partial_\alpha\psi^\nu\right)\eta_{\mu\nu}, \tag{2.71}$$

where the conjugate Majorana spinor is defined as

$$\overline{\psi} \equiv \psi^{\mathrm{t}}\rho^0 = (-\mathrm{i}\psi_+\ \mathrm{i}\psi_-). \tag{2.72}$$

Using

$$-\mathrm{i}\overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi^\nu = -\mathrm{i}\psi_+^\mu \partial_- \psi_+^\nu - \mathrm{i}\psi_-^\mu \partial_+ \psi_-^\nu \,; \qquad\qquad \partial_\pm \equiv \partial_0 - \partial_1 \,, \tag{2.73}$$

we can rewrite the action (Eq. (2.71)) as

$$S = \frac{1}{4\pi\alpha'} \int \mathrm{d}\sigma\, \mathrm{d}\tau \left(\partial_+ X^\mu \partial_- X^\nu + \mathrm{i}\psi_+^\mu \partial_- \psi_+^\nu + \mathrm{i}\psi_-^\mu \partial_+ \psi_-^\nu\right)\eta_{\mu\nu} \,. \tag{2.74}$$

The equations of motion and boundary conditions for X^μ are the same as for the bosonic string. To find the equations of motion for the fermionic fields $\psi_\pm$, we vary them as $\psi_\pm \to \psi_\pm + \delta\psi_\pm$. Let us consider the open string so $\sigma \in [0, \pi]$. After integrating by parts so all derivatives are on $\psi_\pm$ instead of $\delta\psi_\pm$, we find

$$\begin{aligned}\delta S = \frac{\mathrm{i}}{2\pi\alpha'} \int_{-\infty}^{\infty} \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma \left(\delta\psi_+^\mu \partial_- \psi_+^\nu + \delta\psi_-^\mu \partial_+ \psi_-^\nu\right)\eta_{\mu\nu} \\ + \frac{\mathrm{i}}{4\pi\alpha'} \int_{-\infty}^{\infty} \mathrm{d}\tau\, \eta_{\mu\nu} \Big[\psi_-^\mu \delta\psi_-^\nu - \psi_+^\mu \delta\psi_+^\nu\Big]_{\sigma=0}^{\sigma=\pi} .\end{aligned} \tag{2.75}$$

The first line leads to the equation of motion:

$$\partial_\pm \psi_\mp^\mu = 0 \,. \tag{2.76}$$

The boundary term can be made to vanish by imposing $\psi_+^\mu = \pm\psi_-^\mu$ at both $\sigma = 0$ and $\sigma = \pi$. The boundary conditions need to be imposed separately at each end because the two endpoints of the string are out of causal contact so we can't use a boundary condition which mixes them [18]. If we impose the *same* boundary condition at both $\sigma = 0$ and $\sigma = \pi$ *e.g.*

$$\psi_+^\mu(0, \tau) = \psi_-^\mu(0, \tau) \,; \qquad\qquad \psi_+^\mu(\pi, \tau) = \psi_-^\mu(\pi, \tau) \,, \tag{2.77}$$

then Eq. (2.76) can be solved by setting $\psi_\pm^\mu$ to be any *periodic* functions of $\tau \pm \sigma$ with period 2π. Fermions satisfying these boundary conditions are in the *Ramond sector*, and they admit the mode expansion

$$\psi_\mp^\mu(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbf{Z}} d_{\mp,n}^\mu \, \mathrm{e}^{-\mathrm{i}n\pi(\tau \mp \sigma)} \,. \tag{2.78}$$

Alternatively, we can impose opposite boundary conditions for the two endpoints, *e.g.*

$$\psi_+^\mu(0, \tau) = \psi_-^\mu(0, \tau) \,; \qquad\qquad \psi_+^\mu(\pi, \tau) = -\psi_-^\mu(\pi, \tau) \,, \tag{2.79}$$

then Eq. (2.76) can be solved by setting $\psi_\pm^\mu$ to be any *anti-periodic* functions of $\tau \pm \sigma$ with period 2π. Fermions satisfying these boundary conditions are said to be in the *Neveu-Schwarz sector* and they admit the half-integer mode expansion [32]

$$\psi_\mp^\mu(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{r \in \mathbf{Z} + \frac{1}{2}} b_{\mp,r}^\mu \, \mathrm{e}^{-\mathrm{i}r\pi(\tau \mp \sigma)} \,. \tag{2.80}$$

Note that the statements about periodicity and anti-periodicity only apply to the (σ, τ) coordinate system because of the way spinors transform under diffeomorphisms.

Unlike our discussion of the bosonic string, we have not written the superstring action in a reparametrization-invariant way, although it is certainly possible to do so by coupling Eq. (2.71) to two-dimensional supergravity [56, 57] (pedagogical treatments are in *e.g.* section 5.8 of [32] or section 6.1 of [18]).

As we did in the bosonic case, let us take our action Eq. (2.74) in which the worldsheet has been split into chiral components and switch to a Euclidean signature as we did for the bosonic string by putting $\tau \to -\mathrm{i}\tau$. Let us use the complex coordinates defined in Eq. (2.28). The equations of motion for $\psi_\pm$ (Eq. (2.76)) become $\partial\psi_-^\mu = \overline{\partial}\psi_+^\mu = 0$, *i.e.* ψ_+ is holomorphic and ψ_- is anti-holomorphic. Rescaling the $\psi_\pm^\mu$'s so they're appropriately normalized, the action Eq. (2.74) then becomes [30]

$$S_{\mathrm{sc}} = -\frac{1}{2\pi\alpha'}\int \mathrm{d}z\,\mathrm{d}\overline{z}\,\left(\partial X^\mu \overline{\partial} X^\nu - \psi_+^\mu \overline{\partial}\psi_+^\nu - \psi_-^\mu \partial \psi_-^\nu\right)\eta_{\mu\nu}\,. \tag{2.81}$$

This form of the superstring action is called *superconformal gauge*. Now, the bosonic string action in conformal gauge in complex coordinates (Eq. (2.37)) was invariant under holomorphic changes of coordinates (Eq. (2.38)); for this to be a symmetry of the action S_{sc} we need $\psi_\pm^\mu$ to change under coordinate transformations as

$$\psi_+^{\mu'}(z') = \left(\frac{\partial z'}{\partial z}\right)^{-\frac{1}{2}}\psi_+^\mu(z)\,; \qquad\qquad \psi_-^{\mu'}(\overline{z}') = \left(\frac{\partial \overline{z}'}{\partial \overline{z}}\right)^{-\frac{1}{2}}\psi_+^\mu(\overline{z})\,. \tag{2.82}$$

Each component of the worldsheet fermion in superconformal gauge is therefore an (anti-) holomorphic section of a complex line bundle which transforms similarly to components of a tensor (Eq. (2.46)) but with $n = -1/2$; in particular the square (in the sense of the Picard group) of the line bundle which ψ_+^μ is a section of is the dual of the canonical bundle K^{-1}.

Then we can change coordinates so the worldsheet is parametrized as the upper-half plane, as we did for the bosonic string; we can achieve this by setting $z \to z' = \mathrm{e}^{\mathrm{i}z}$, $\overline{z} \to \overline{z}' = \mathrm{e}^{-\mathrm{i}\overline{z}}$. The form of the worldsheet Lagrangian is unchanged, however, ψ_+ in the new coordinates includes a factor of $(\mathrm{i}z')^{-\frac{1}{2}}$ because of its transformation properties. This changes by a factor of -1 when moved in a closed cycle around $z' = 0$; this means that the fermions in the Ramond sector which were periodic on the strip become *anti-periodic* on the complex plane, while fermions in the Neveu-Schwarz sector which were anti-periodic on the strip become *periodic* on the plane.

Just as with the bosonic string, we don't have to restrict ourselves to worldsheets with the topology of a strip; we can add handles and cut out discs to get any oriented Riemann surface. The only qualification is that we need to be able to define spinors globally, *i.e.* we need the surfaces to admit line bundles whose transition functions are of the form in Eq. (2.82) (called *spin bundles*). In fact, every Riemann surface admits spin bundles [46]. Any spin bundle has a *spin structure*: the transition functions allow spinors which pick up a phase of $\mathrm{e}^{\mathrm{i}\pi}$ when transported around a closed curve as well as spinors which transform trivially; the spin structure is equivalent to a prescription for how a spinor transforms when transported around any closed cycle. It can be specified by listing the phases ± 1 associated to each of the $2g$ homology cycles in any canonical basis, and each of these is independent, therefore there are 2^{2g} spin structures on a genus g Riemann surface [35].

2.1.5 Super-Riemann surfaces

Is is useful to formulate superstring perturbation theory in the language of *super-Riemann surfaces* [58, 59, 60].

The superstring action in superconformal gauge (Eq. (2.81)) can also be written using a superfield formalism: we supplement the bosonic complex coordinates $(z, \overline{z})$ with anti-commuting (or 'Grassmann') coordinates $(\theta, \overline{\theta})$; we can write the coordinates in pairs as $\mathbf{z} = z|\theta$; $\overline{\mathbf{z}} = \overline{z}|\overline{\theta}$. The two worldsheet fields X^μ and $\psi^\mu_\pm$ can then be grouped together (along with an auxiliary field) into a *superfield* $\mathbf{X}^\mu$, which can be Taylor expanded in θ, $\overline{\theta}$ as:

$$\mathbf{X}^\mu(\mathbf{z}, \overline{\mathbf{z}}) = X^\mu(z, \overline{z}) + \theta\psi_+(z, \overline{z}) + \overline{\theta}\psi_-(z, \overline{z}) + \theta\overline{\theta}N^\mu(z, \overline{z}) \,, \tag{2.83}$$

where the Taylor series is exact because $\theta^2 = \overline{\theta}^2 = 0$.

We can introduce *superderivatives* D_θ and $D_{\overline{\theta}}$ defined by

$$D_\theta \equiv \partial_\theta + \theta\partial_z \,; \qquad\qquad \overline{D}_\theta = \partial_{\overline{\theta}} + \overline{\theta}\partial_{\overline{z}} \,, \tag{2.84}$$

where the derivative with respect to a Grassmann coordinate θ, ∂_θ, is also anti-commuting. It effectively selects the coefficient of θ, once θ has been anti-commuted to the left-hand-side of the expression. D_θ and $\overline{D}_\theta$ act on the superfield Eq. (2.83) to give

$$D_\theta\mathbf{X}^\mu = \psi^\mu_+ + \overline{\theta}N^\mu + \theta\,\partial X^\mu + \theta\overline{\theta}\partial\psi^\mu_- \,, \tag{2.85}$$

$$\overline{D}_\theta\mathbf{X}^\mu = \psi^\mu_- - \theta N^\mu + \overline{\theta}\,\overline{\partial}X^\mu + \overline{\theta}\theta\overline{\partial}\psi^\mu_+ \,. \tag{2.86}$$

We can introduce the *Berezin integral* for integration of a superfield over fermionic coordinates:

$$\int [\mathrm{d}^n z|\mathrm{d}\theta\,\mathrm{d}\overline{\theta}]\big(f_{00}(z_i) + \theta f_{10}(z_i) + \overline{\theta}f_{01}(z_i) + \overline{\theta}\theta f_{11}(z_i)\big) = \int \mathrm{d}^n z\, f_{11}(z_i) \,, \tag{2.87}$$

i.e. the integration over $\mathrm{d}\theta\,\mathrm{d}\overline{\theta}$ picks out the coefficient of $\overline{\theta}\theta$. But the coefficient of $\overline{\theta}\theta$ in $\overline{D}_\theta\mathbf{X}^\mu D_\theta\mathbf{X}^\nu\eta_{\mu\nu}$ is equal to the z-integrand of the string action in superconformal gauge (Eq. (2.81)) with an auxiliary term, *i.e.* we have

$$\int [\mathrm{d}^2 z|\mathrm{d}^2\theta]\,\overline{D}_\theta\mathbf{X}^\mu D_\theta\mathbf{X}_\mu = \int \mathrm{d}^2 z\,(\overline{\partial}X^\mu\partial X^\nu - \psi^\mu_+\overline{\partial}\psi^\nu_+ - \psi^\mu_-\partial\psi^\nu_- + N^\mu N^\nu)\,\eta_{\mu\nu} \,. \tag{2.88}$$

but N^μ doesn't interact with any of the other fields so we can set it to 0 with its equation of motion [17]. The string action Eq. (2.81) can then be rewritten as

$$S_{\text{sc}} = -\frac{1}{2\pi\alpha'}\int [\mathrm{d}^2 z|\mathrm{d}^2\theta]\,\overline{D}_\theta\mathbf{X}^\mu D_\theta\mathbf{X}_\mu \,. \tag{2.89}$$

The equation of motion for $\mathbf{X}^\mu$ in superconformal coordinates is $D_\theta \overline{D}_\theta \mathbf{X}^\mu = 0$, whose general solution is $\mathbf{X}^\mu(\mathbf{z}, \overline{\mathbf{z}}) = X_\mathrm{L}(z) + \mathrm{i}\theta\psi_+(z) + X_\mathrm{R}(\overline{z}) + \mathrm{i}\overline{\theta}\psi_-(\overline{z})$, *i.e.* it splits into analytic and anti-analytic superfields [61]

$$\mathbf{X}^\mu(\mathbf{z}, \overline{\mathbf{z}}) = \mathbf{X}^\mu_\mathrm{L}(\mathbf{z}) + \mathbf{X}^\mu_\mathrm{R}(\overline{\mathbf{z}})\,; \qquad \mathbf{X}^\mu_\mathrm{L}(\mathbf{z}) = X^\mu_\mathrm{L}(z) + \mathrm{i}\theta\psi^\mu_+(z)\,; \tag{2.90}$$

$$\mathbf{X}^\mu_\mathrm{R}(\overline{\mathbf{z}}) = X^\mu_\mathrm{R}(\overline{z}) + \mathrm{i}\overline{\theta}\psi^\mu_-(\overline{z})\,. \tag{2.91}$$

We have seen that because the bosonic string action in conformal gauge (Eq. (2.37)) is unchanged by holomorphic changes of coordinates, it is well-defined on a Riemann surface. By analogy, we want to find a class of objects with potentially more complicated topology where the superstring action in superconformal gauge (Eq. (2.89)) is manifestly well-defined. It turns out that the correct setting is *super-Riemann surfaces* (SRS).

A SRS is a type of $1|1$-dimensional complex supermanifold, *i.e.* it can be described locally by one commuting complex coordinate and one anticommuting complex coordinate; typically we write the coordinates of a point in some chart in the form $z|\theta$. On a super-Riemann surface Σ, there is some additional structure: the tangent bundle contains a rank $0|1$ sub-bundle $\mathcal{D}$, such that for any section D of $\mathcal{D}$, at every point on Σ, $D^2 = \frac{1}{2}\{D, D\}$ is linearly independent of D [60]. Concretely, if $z|\theta$ are local coordinates on Σ, then $\{\partial_z, \partial_\theta\}$ is a local basis for $T\Sigma$ and we can define

$$D_\theta \equiv \partial_\theta + \theta\partial_z\,. \tag{2.92}$$

Since we can expand any function f as $f(z, \theta) = g(z) + \theta h(z)$, we have

$$D_\theta D_\theta f = \partial_z f \tag{2.93}$$

so $D_\theta^2 = \partial_z$, which is linearly independent of D_θ. In fact, this choice of D is quite general: for any section D which satisfies $D^2 = D$ nowhere, we can change co-ordinates such that D is of the form D_θ (see section 2.1 of [60]).

Taking advantage of this, we can choose only to use coordinates in which $\mathcal{D}$ is spanned by a section of the form Eq. (2.92); we call these *superconformal coordinates*. We can find an analogue of the Cauchy-Riemann equations which the transition function must satisfy to be superconformal. Suppose that $\hat{z}|\hat{\theta} \mapsto z|\theta$ is some transition function to change co-ordinates between overlapping charts. Let f be a function on the intersection of the charts, then by the chain rule we have $D_{\hat{\theta}} f = (D_{\hat{\theta}}\theta)\partial_\theta f + (D_{\hat{\theta}} z)\partial_z f$. We need this to be proportional to $D_\theta f$; this will hold if

$$D_{\hat{\theta}} z = \theta D_{\hat{\theta}}\theta\,, \tag{2.94}$$

which is the required condition for a change of coordinates to be superconformal.

If $\widehat{z}|\widehat{\theta}$, $z|\theta$ are two superconformal charts on a SRS, then the two volume forms $[\mathrm{d}\widehat{z}|\mathrm{d}\widehat{\theta}]D_{\widehat{\theta}} f$ and $[\mathrm{d}z|\mathrm{d}\theta]D_\theta f$ are identical, *i.e.* their Berezin integrals are the same (see *e.g.* section 2.4 of [60]). From this it follows that the superstring action Eq. (2.89) is well-defined globally on a SRS, where the integration variables are any local choice of superconformal coordinates.

2.1.6 Super-projective transformations

The simplest compact super-Riemann surface is the super-Riemann sphere $\mathbf{CP}^{1|1}$. This can be defined in terms of homogeneous coordinates in $\mathbf{C}^{2|1}$ by the equivalence relation $(z_1, z_2|\zeta) \sim (\lambda z_1, \lambda z_2|\lambda\theta)$ for non-zero complex λ, where the bosonic coordinates z_1 and z_2 are not allowed to vanish simultaneously. It can be covered by two coordinate charts: for $z_2 \neq 0$ we can use $z|\theta \equiv \frac{z_1}{z_2}|\frac{\zeta}{z_2}$ and for $z_1 \neq 0$ we can use $w|\psi \equiv \frac{-z_2}{z_1}|\frac{\zeta}{z_1}$. Both coordinate charts are superconformal, *i.e.* Eq. (2.94) is satisfied by the transition function between them.

The transition function between the two charts is an automorphism of $\mathbf{CP}^{1|1}$; other automorphism are generated by super-conformal versions of translations and dilatations given by

$$z|\theta \mapsto z - a - \theta\alpha|\theta - \alpha \, ; z|\theta \mapsto \lambda^2 z|\lambda\theta \, . \tag{2.95}$$

From these we can generate the full group of automorphisms, the orthosymplectic group OSp(1|2) [60] (also called OSp(1, 1) in *e.g.* [39]), which can be realised by matrices of the form

$$\mathbf{S} = \left(\begin{array}{cc|c} a & b & \alpha \\ c & d & \beta \\ \hline \gamma & \delta & e \end{array}\right) \tag{2.96}$$

where the 5 bosonic and 4 fermionic variables are subject to the 2 fermionic and 2 bosonic constraints,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -\delta \\ \gamma \end{pmatrix} \, ; \qquad ad - bc - \alpha\beta = 1 \, ; \qquad e = 1 - \alpha\beta \, , \tag{2.97}$$

so the group has dimension 3|2.

If we define a skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the homogeneous co-ordinates by

$$\langle z, y \rangle = z_1 y_2 - z_2 y_1 - \theta\psi \tag{2.98}$$

then OSp(1|2) can be characterized as the subgroup of GL(2|1) which preserves $\langle \cdot, \cdot \rangle$ [60].

We can find an OSp(1|2) matrix taking $\mathbf{u} = (u_1, u_2|\theta)$ and $\mathbf{v} = (v_1, v_2|\phi)$ to points equivalent to $(0, 1|0)$ and $(1, 0|0)$ respectively; one such matrix is

$$\mathbf{\Gamma_{uv}} = \frac{1}{\sqrt{\langle \mathbf{u}, \mathbf{v} \rangle}} \left(\begin{array}{cc|c} u_2 & -u_1 & \theta \\ v_2 & -v_1 & \phi \\ \hline \frac{u_2\phi - v_2\theta}{\sqrt{\langle \mathbf{u}, \mathbf{v} \rangle}} & \frac{v_1\theta - u_1\phi}{\sqrt{\langle \mathbf{u}, \mathbf{v} \rangle}} & \sqrt{\langle \mathbf{u}, \mathbf{v} \rangle} - \frac{\theta\phi}{\sqrt{\langle \mathbf{u}, \mathbf{v} \rangle}} \end{array}\right) . \tag{2.99}$$

We have one bosonic degree of freedom remaining; we can stipulate that a point $\mathbf{w} = (w_1, w_2|\omega)$ is mapped to a point equivalent to $(1, 1|\Theta_{\mathbf{uwv}})$ where there is no freedom in choosing the fermionic co-ordinate, which is therefore a super-projective invariant of $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$. The image of $\mathbf{w}$ under $\mathbf{\Gamma_{uv}}$ is

$$\mathbf{\Gamma_{uv}w} = \frac{1}{\sqrt{\langle \mathbf{u}, \mathbf{v}\rangle}}\Big(\langle \mathbf{w}, \mathbf{u}\rangle, \langle \mathbf{w}, \mathbf{v}\rangle \Big| \frac{\theta\langle \mathbf{v}, \mathbf{w}\rangle + \phi\langle \mathbf{w}, \mathbf{u}\rangle + \omega\langle \mathbf{u}, \mathbf{v}\rangle + \omega\theta\phi}{\sqrt{\langle \mathbf{u}, \mathbf{v}\rangle}}\Big). \tag{2.100}$$

A general dilatation of the superconformal co-ordinates corresponds to the OSp(1|2) matrix

$$\mathbf{P}(y) = \left(\begin{array}{cc|c} y^{\frac{1}{2}} & 0 & 0 \\ 0 & y^{-\frac{1}{2}} & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \tag{2.101}$$

which leaves invariant the points $(0, 1|0)$ and $(1, 0|0)$. We may use a transformation like this to scale the bosonic coordinates of $\mathbf{\Gamma_{uv}w}$ as desired, obtaining

$$\mathbf{P}\Big(\frac{\langle \mathbf{w}, \mathbf{v}\rangle}{\langle \mathbf{w}, \mathbf{u}\rangle}\Big)\mathbf{\Gamma_{uv}w} \sim \Big(1, 1\Big| \frac{\theta\langle \mathbf{w}, \mathbf{v}\rangle + \omega\langle \mathbf{v}, \mathbf{u}\rangle + \phi\langle \mathbf{u}, \mathbf{w}\rangle + \theta\omega\phi}{\sqrt{\langle \mathbf{v}, \mathbf{u}\rangle\langle \mathbf{u}, \mathbf{w}\rangle\langle \mathbf{w}, \mathbf{v}\rangle}}\Big) \tag{2.102}$$

giving us an explicit expression for the odd super-projective invariant $\Theta_{\mathbf{z_1 z_2 z_3}}$:

$$\Theta_{\mathbf{z_1 z_2 z_3}} = \frac{\zeta_1\langle \mathbf{z}_2, \mathbf{z}_3\rangle + \zeta_2\langle \mathbf{z}_3, \mathbf{z}_1\rangle + \zeta_3\langle \mathbf{z}_1, \mathbf{z}_2\rangle + \zeta_1\,\zeta_2\,\zeta_3}{\sqrt{\langle \mathbf{z}_1, \mathbf{z}_2\rangle\langle \mathbf{z}_2, \mathbf{z}_3\rangle\langle \mathbf{z}_3, \mathbf{z}_1\rangle}}, \tag{2.103}$$

where $\mathbf{z}_i = z_i|\zeta_i$, as in Eq. (3.222) of [39].

2.2 Quantization

In this section we discuss the quantization of the bosonic strings and superstrings we discussed in section 2.1. We use the 'first-quantized' path integral formalism in which we treat the embedding functions X^μ and the worldsheet metric $h_{\alpha\beta}$ as integration variables. We recall facts about conformal field theory (CFT) and show that the BRST quantization procedure can be implemented by coupling the worldsheet metric to a ghost CFT.

Use has been made of the lecture notes by Friedan [48, 58], Dixon [30], Alvarez [47] and Tong [21], the textbook by Polchinski [62] and classic papers by Belavin, Polyakov and Zamolodchikov [63], Friedan, Martinec and Shenker [61] and D'Hoker and Phong [39]. Quantization of the worldsheet action was first carried out by Polyakov for the bosonic string in [29] and the superstring in [55].

2.2.1 The gauge symmetry of bosonic string theory

We've seen in section 2.1.2 that the bosonic string is described by an action (Eq. (2.6)) which is a functional of X^μ, which describes the embedding of the string worldsheet into spacetime, and $h_{\alpha\beta}$, the metric on the worldsheet. In section 2.1.3 we discussed how the action can be generalized to worldsheets with handles and additional boundaries. As in any quantum theory, we can compute observables by evaluating a path integral over all possible 'histories', which in this case means taking some fixed abstract two-dimensional manifold Σ (with a fixed choice of coordinates), and then integrating over both the space $\mathscr{E}$ of embedding functions X^μ and over the space $\mathscr{M}_g$ of worldsheet metrics $h_{\alpha\beta}$. The path integral includes a sum over all possible topologies of the worldsheet. In the case of closed string worldsheets, this is equivalent to a sum over genera (*i.e.* number of handles) but there are more topologies to sum over when boundaries are permitted.

We have seen that the action Eq. (2.6) is invariant under both the group of reparametrizations of the surface $\mathscr{D}(\Sigma)$ (Eq. (2.13)) and the group of Weyl rescalings $\mathscr{C}(\Sigma)$ (Eq. (2.11)). This means that there is the possibility of over-counting: one pair of functions $(X^\mu; h_{\alpha\beta})$ on Σ may be equivalent to a different pair of functions $(\tilde{X}^\mu; \tilde{h}_{\alpha\beta})$ on Σ up to the action of a reparametrization of Σ; in this case we want to ensure that only one from each equivalence class is counted. Moreover, we want to ensure that if two metric on a surface are related by a Weyl scaling then only one of them is counted. The gauge group is then the semidirect product $G = \mathscr{C}(\Sigma) \ltimes \mathscr{D}(\Sigma)$ [64]. The integration has to be carried out over the space $\mathscr{E} \times \mathscr{M}_g/(\mathscr{C}(\Sigma) \ltimes \mathscr{D}(\Sigma))$.

Diffeomorphisms generated by vector fields ξ^α constitute only the identity component $\mathscr{D}_0(\Sigma)$ of the full group of diffeomorphisms. $\mathscr{D}_0(\Sigma)$ is a normal subgroup of $\mathscr{D}(\Sigma)$ and the quotient is called the *mapping class group* ($\mathscr{D}(\Sigma)/\mathscr{D}_0(\Sigma) = \mathrm{MCG}$), which is a discrete group [31, 52]. The quotient of the space of genus-g metrics $\mathscr{M}_g$ by $\mathscr{C}(\Sigma) \ltimes \mathscr{D}_0(\Sigma)$ is called *Teichmüller space* $\mathcal{T}_g$. Teichmüller space over-counts since we are only interested in inequivalent surfaces, because we have quotiented only by the identity component of the diffeomorphism group $\mathscr{D}_0$; really we should have divided $\mathscr{M}_g$ by the full group of diffeomorphisms which would leave us with *moduli space* $\mathcal{M}_g = \mathcal{T}_g/\mathrm{MCG}$ instead of Teichmüller space. Roughly, moduli space is the space of complex structures on a surface with a given topology [31, 39].

The integration measures on $\mathscr{E}$ and $\mathscr{M}_g$ were constructed by Polyakov in [29], with developments in [47] for open string worldsheets; more details are given in [48, 33, 39]. The measures are constrained by 'ultralocality', *i.e.* independence of the derivatives of $h_{\alpha\beta}$ and δX^μ. We will not look at the construction of the measure but just make use of the results.

The path integral for bosonic string theory is written, then, as an integral over $\mathscr{E} \times \mathscr{M}_g$ divided schematically by the 'volume' of the gauge group:

$$Z = \int \mathcal{D}h_{\alpha\beta}\, \mathcal{D}X^\mu \, \frac{1}{\mathrm{Vol}(\mathscr{C}(\Sigma) \ltimes \mathscr{D}(\Sigma))} e^{-S_{\mathrm{bos}}[X,h]} \tag{2.104}$$

We can denote a general gauge transformation as ζ and we can write the corresponding change of metric as $h_{\alpha\beta} \mapsto h^\zeta_{\alpha\beta}$. We need to choose a particular gauge for our computations, such as the conformal gauge (Eq. (2.15)), by specifying its functional form; in general we call our arbitrary chosen metric the *fiducial* metric $\hat{h}_{\alpha\beta}$.

The fiducial metric $\hat{h}_{\alpha\beta}$ is some unique way of choosing a representative for each class of physically equivalent metrics, so integrating over the space of all fiducial metrics is the same as integrating over the space of all physically distinct configurations.

Formally, we can gauge fix to get the fiducial metric by inserting a functional Dirac δ-function. This would change the value of the path integral so we need to include another term to cancel it. If we were able to integrate the δ-function over a set of physically equivalent metrics then we would be able to insert a factor of unity, but it is not clear how to perform this integration in general; we actually have to perform the integral over the gauge group:

$$\int \mathcal{D}\zeta\, \delta(h_{\alpha\beta} - \hat{h}^{\zeta}_{\alpha\beta}) = \frac{1}{\Delta_{\text{FP}}[h]} \tag{2.105}$$

which defines $\Delta_{\text{FP}}[h]$, the *Faddeev-Popov determinant* evaluated for a metric $h_{\alpha\beta}$.

For any gauge transformation ζ we have $\Delta_{\text{FP}}[h] = \Delta_{\text{FP}}[h^{\zeta}]$ so Δ_{FP} is gauge-invariant; this follows from the gauge-invariance of the integration measure and of the δ function. We can use this to insert a factor of unity, $1 = \Delta_{\text{FP}}[h] \int \mathcal{D}\zeta\, \delta(h_{\alpha\beta} - \hat{h}^{\zeta}_{\alpha\beta})$ in the path integral

$$Z = \int \mathcal{D}\zeta\, \mathcal{D}h\, \mathcal{D}X^{\mu}\, \Delta_{\text{FP}}[h]\delta(h_{\alpha\beta} - \hat{h}^{\zeta}_{\alpha\beta}) \frac{1}{\text{Vol}(\mathscr{C}(\Sigma) \ltimes \mathscr{D}(\Sigma))} e^{-S_{\text{bos}}}[X, h] \tag{2.106}$$

so we can perform the $h_{\alpha\beta}$ integration and the δ function replaces each instance of $h_{\alpha\beta}$ with $\hat{h}^{\zeta}_{\alpha\beta}$:

$$Z = \int \mathcal{D}\zeta\, \mathcal{D}X^{\mu}\, \Delta_{\text{FP}}[\hat{h}^{\zeta}]\, \frac{1}{\text{Vol}} e^{-S_{\text{bos}[X,\hat{h}^{\zeta}]}} \tag{2.107}$$

but since both Δ_{FP} and S_{bos} are gauge-invariant we can replace $\hat{h}^{\zeta}_{\alpha\beta}$ with $\hat{h}_{\alpha\beta}$ everywhere so only the fiducial metric appears

$$Z = \int \mathcal{D}\zeta\, \mathcal{D}X^{\mu}\, \Delta_{\text{FP}}[\hat{h}]\, \frac{1}{\text{Vol}(\mathscr{C}(\Sigma) \ltimes \mathscr{D}(\Sigma))} e^{-S_{\text{bos}}[X,\hat{h}]}. \tag{2.108}$$

Note that nothing in the integrand now depends on the gauge transformation ζ, so the $\int \mathcal{D}\zeta$ path integral is now just a constant factor corresponding to the number of physically equivalent configurations; it cancels $1/\text{Vol}(\mathscr{C}(\Sigma) \ltimes \mathscr{D}(\Sigma))$ and we are left with

$$Z = \int \mathcal{D}X^{\mu}\, \Delta_{\text{FP}}[\hat{h}] e^{-S_{\text{bos}}[X,\hat{h}]}. \tag{2.109}$$

This is an integral over physically distinct configurations with a canonical choice of fiducial metric $\hat{h}$. It is weighted appropriately by Δ_{FP}, which it wouldn't be if we had naïvely specified the functional form of the metric and integrate over field configurations. It still remains to compute Δ_{FP}.

2.2.2 The Faddeev-Popov determinant

The next step is to compute the expression in Eq. (2.105) for $\Delta^{-1}_{\text{FP}}[h]$. Considering only infinitesimal gauge transformations ζ close to the identity, we can write a first-order expansion $\hat{g}^{\zeta}_{\alpha\beta} \approx \hat{h}_{\alpha\beta} + \delta\hat{h}_{\alpha\beta}$ where $\delta\hat{h}_{\alpha\beta}$ can include both infinitesimal diffeomorphisms (Eq. (2.13)) and Weyl transformations (Eq. (2.11)) of the infinitesimal form $h_{\alpha\beta} \to (1 + 2\omega)h_{\alpha\beta}$, giving

$$\delta\hat{h}_{\alpha\beta} = 2\omega\hat{h}_{\alpha\beta} + \nabla_{\alpha}\xi_{\beta} + \nabla_{\beta}\xi_{\alpha}\,. \tag{2.110}$$

Inserting this expression for $h_{\alpha\beta} - \hat{h}^{\zeta}_{\alpha\beta}$ in the δ function in Eq. (2.105), we can write the integral over all gauge transformations as a functional integral over all infinitesimal diffeomorphisms ξ^α and Weyl transformations ω, then

$$\Delta_{\mathrm{FP}}^{-1}[\hat{h}] = \int \mathcal{D}\omega\, \mathcal{D}\xi^\alpha \delta(2\omega\hat{h}_{\alpha\beta} + \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha)\,, \tag{2.111}$$

This is like integrating over the Lie algebra of a Lie group except the gauge group in this case is an infinite-dimensional space of functions.

We may replace the δ-function with its integral form obtaining

$$\Delta_{\mathrm{FP}}^{-1}[g] = \int \mathcal{D}\omega\, \mathcal{D}\xi^\alpha\, \mathcal{D}\beta^{\alpha\beta}\, \exp\left(2\pi\mathrm{i}\int d^2\sigma\, \sqrt{-\hat{h}}\, \beta^{\alpha\beta}(2\omega\hat{h}_{\alpha\beta} + \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha)\right)\,, \tag{2.112}$$

where $\beta^{\alpha\beta}$ is a rank 2 tensor on the worldsheet which multiplies $\hat{h}_{\alpha\beta}$ so we can assume it is symmetric. Performing the $\mathcal{D}\omega$ integral we obtain a δ function

$$\int \mathcal{D}\omega \exp\left(2\pi\mathrm{i}\int d^2\sigma\, \sqrt{-\hat{h}}\, \beta^{\alpha\beta}\hat{h}_{\alpha\beta}\, 2\omega\right) = \delta(2\beta^{\alpha\beta}\hat{h}_{\alpha\beta})\,, \tag{2.113}$$

which fixes $\beta^{\alpha\beta}\hat{h}_{\alpha\beta} = 0$ so we take $\beta^{\alpha\beta}$ to be traceless. We then replace the $\beta^{\alpha\beta}$ integral with one over all symmetric, traceless $\tilde{\beta}^{\alpha\beta}$:

$$\Delta_{\mathrm{FP}}^{-1}[\hat{h}] = \int \mathcal{D}\omega\, \mathcal{D}\tilde{\beta}^{\alpha\beta}\, \exp\left(4\pi\mathrm{i}\int d^2\sigma\sqrt{-\hat{h}}\, \tilde{\beta}^{\alpha\beta}\, \nabla_\alpha v_\beta\right) \tag{2.114}$$

where we have used the symmetry of $\tilde{\beta}^{\alpha\beta}$ to write $\tilde{\beta}^{\alpha\beta}(\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha) = 2\beta^{\alpha\beta}\nabla_\alpha\xi_\beta$.

We have an expression for $\Delta_{\mathrm{FP}}^{-1}[h]$ but we need to invert it to get $\Delta_{\mathrm{FP}}[h]$. This can be done with the simple trick of changing the integration variables $\tilde{\beta}^{\alpha\beta}$ and ξ^α from commuting to anticommuting *ghost* fields: let

$$\tilde{\beta}_{\alpha\beta} \to b_{\alpha\beta}\,; \qquad\qquad \xi^\alpha \to c^\alpha\,, \tag{2.115}$$

giving the path integral

$$\Delta_{\mathrm{FP}}[h] = \int \mathcal{D}b_{\alpha\beta}\, \mathcal{D}c^\alpha\, \exp(\mathrm{i}S_{\mathrm{gh}}[b, c, \hat{h}]) \tag{2.116}$$

where

$$S_{\mathrm{gh}}[b, c, h] = \frac{1}{2\pi}\int d^2\sigma\, \sqrt{-h}\, b_{\alpha\beta}\nabla^\alpha c^\beta. \tag{2.117}$$

In conformal gauge (Eq. (2.15)) this is particularly simple:

$$S_{\mathrm{gh}} = \frac{1}{2\pi}\int d^2z\, \left(b_{zz}\overline{\partial}\, c^z + b_{\overline{z}\overline{z}}\partial\, c^{\overline{z}}\right)\,, \tag{2.118}$$

which doesn't depend on the Weyl scaling factor ω.

Then the Polyakov path integral is given as

$$Z[\hat{h}] = \int \mathcal{D}X\mathcal{D}b_{\alpha\beta}\,\mathcal{D}c^{\alpha}e^{-S_{\rm bos}[\hat{h}]-S_{\rm ghost}[\hat{h},b,c]}. \tag{2.119}$$

RNS superstrings

For superstrings, the situation is analogous. The full worldsheet action now depends not only on the metric $h_{\alpha\beta}$ but also on its superpartner, a spin-$\frac{3}{2}$ gravitino χ_α (see *e.g.* §3.1 of [39], §6.1 of [18], §5.8 of [32] or §4.1 of [61]). To write the worldsheet action in the form of Eq. (2.89) requires choosing a gauge slice for both the metric and the gravitino; the gauge slice we have chosen is called the superconformal gauge. Just as the bosonic string had a residual Weyl symmetry (Eq. (2.11)) even after fixing conformal gauge (Eq. (2.37)); so the RNS superstring has a residual superconformal symmetry even after fixing superconformal gauge. This can be seen from the form of the action Eq. (2.89) which is invariant under superconformal changes of coordinates.

Just as the path integral for bosonic string theory can be reduced from a redundant integral over all worldsheet metrics $h_{\alpha\beta}$ to an integral over the finite-dimensional moduli space of Riemann surfaces, so the path integral for RNS superstrings can be reduced from a redundant path integral over worldsheet metrics $h_{\alpha\beta}$ and gravitons χ_α to an integral over the finite-dimensional space of SRSs with a given topology, called *super-moduli space* $\mathfrak{M}$. Super-moduli space is a complex supermanifold in its own right (to be more precise, it's an orbifold because the mapping class group has fixed points). The super-moduli space of compact SRSs of genus $g > 1$, $\mathfrak{M}_g$, has complex dimension $3g-3|2g-2$ [65].

We can replace the integral over all metrics and gravitinos by an integral over a gauge slice multiplied by a Faddeev-Popov determinant, as for the bosonic string, but for the superstring the requisite ghost fields are superfields. For the analytic sector, let [61]

$$C^z = c^z + \theta\gamma^\theta\,; \qquad\qquad B_{z\theta} = \beta_{z\theta} + \theta b_{zz}\,, \tag{2.120}$$

with $\overline{C}^{\overline{z}}$ and $\overline{B}_{\overline{z}\overline{\theta}}$ defined similarly. Since θ is anti-commuting and so are b_{zz} and c^z, it follows that γ^θ and $\beta_{z\theta}$ must be commuting variables. The ghost action for the superstring becomes

$$S_{\rm gh} = \frac{1}{2\pi}\int [\mathrm{d}^2z|\mathrm{d}^2\theta]\,(B_{z\theta}\overline{D}_\theta C^z + \overline{B}_{\overline{z}\overline{\theta}}D_\theta\overline{C}^{\overline{z}})\,, \tag{2.121}$$

writing this out in terms of the component fields we find

$$S_{\rm gh} = \frac{1}{2\pi}\int \mathrm{d}^2z\,(b_{zz}\overline{\partial}c^z + \beta_{z\theta}\,\overline{\partial}\,\gamma^\theta + \overline{b}_{\overline{z}\overline{z}}\partial\overline{c}^{\overline{z}} + \overline{\beta}_{\overline{z}\overline{\theta}}\,\partial\,\overline{\gamma}^{\overline{\theta}})\,, \tag{2.122}$$

which reduces to the ghost action for the bosonic case (Eq. (2.118)) when we set the β, γ fields to 0.

2.2.3 Conformal Field Theory

As we have seen in section 2.1.2, the classical bosonic string can be described in complex coordinates by an action of the form Eq. (2.37) which is invariant under conformal transformations of the form Eq. (2.38). We want to consider the theory obtained by quantizing this action, *i.e.* by promoting the worldsheet X^μ field theory to a quantum field theory. Note that this is a 'first-quantized' version of string theory [61] in the same sense that quantum mechanics can be conceptualized as a 1-dimensional QFT on the worldline of a particle; we will not discuss the second-quantized theory of strings (known as *string field theory*). Because of the conformal invariance of the action, we use the specialized language of *conformal field theory* (CFT) which is more useful for theories with this symmetry [21].

In the string action we are using Eq. (2.6) there is a worldsheet metric $h^{\alpha\beta}$ which, although it has a less obvious physical meaning than the fields X^μ, must still be integrated over in the path integral. The fundamental objects we will be interested in calculating are expectation values of operators evaluated with the path integral:

$$\left\langle \mathcal{O}_1(z_1,\overline{z}_1),\ldots,\mathcal{O}_n(z_n,\overline{z}_n)\right\rangle \equiv \int \mathcal{D}h^{\alpha\beta}\,\mathcal{D}X^\mu\,\mathrm{e}^{-S_{\mathrm{bos}}}\Big(\mathcal{O}_1(z_1,\overline{z}_1),\ldots,\mathcal{O}_n(z_n,\overline{z}_n)\Big)\,. \quad (2.123)$$

Here the local operators inside the correlation function are understood to be *radially ordered i.e.* $|z_1| > |z_2| > \ldots > |z_n|$ [30]; this is equivalent to τ-ordering and then changing to radial coordinates Eq. (2.45).

We can find the propagator for X^μ. Writing the conformal gauge bosonic string action Eq. (2.6) in cartesian coordinates as $S_{\mathrm{bos}} = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\,\partial_\alpha X^\mu \partial^\alpha X_\mu$, we can use

$$\frac{\delta}{\delta X_\mu(\sigma_1)}\Big[\mathrm{e}^{-S_{\mathrm{bos}}}X^\nu(\sigma_2)\Big] = \mathrm{e}^{-S_{\mathrm{bos}}}\Big(\eta^{\mu\nu}\delta^2(\sigma_1-\sigma_2) + \frac{1}{2\pi\alpha'}\partial_\alpha\partial^\alpha X^\mu(\sigma_1)X^\nu(\sigma_2)\Big)\,, \quad (2.124)$$

along with the fact that the path integral of a functional derivative vanishes to get [21]

$$\langle \partial_\alpha\partial^\alpha X^\mu(\sigma_1)X^\nu(\sigma_2)\rangle = -2\pi\alpha'\eta^{\mu\nu}\delta^2(\sigma_1-\sigma_2)\,, \quad (2.125)$$

which is a partial differential equation for the propagator $\langle X^\mu(\sigma_1), X^\nu(\sigma_2)\rangle$ whose solution on the complex plane is [21]

$$\langle X^\mu(\sigma_1), X^\nu(\sigma_2)\rangle = \alpha' G(\sigma_1,\sigma_2)\eta^{\mu\nu} = -\frac{\alpha'}{2}\log|\sigma_1-\sigma_2|^2\eta^{\mu\nu}\,. \quad (2.126)$$

The scalar propagator in complex coordinates is given by $G(z,w) = -\log 2|z-w|^2 \sim -\log(z-w) - \log(z^*-w^*)$. On a higher-genus Riemann surface the propagator is given in complex coordinates in terms of the prime form (Eq. (2.64)) by

$$G(x,y) = -\log|E(x,y)|^2 + 2\pi\big(\mathrm{Im}\int_y^x \vec{\omega}\big)\cdot(\mathrm{Im}\tau)^{-1}\cdot\big(\mathrm{Im}\int_y^x \vec{\omega}\big)\,. \quad (2.127)$$

Operators evaluated at nearby points can be expressed in terms of the *operator product* *expansion* (OPE) [30]

$$\mathcal{O}_i(z,\overline{z})\mathcal{O}_j(w,\overline{w}) = \sum_k C_{ij}{}^k(z-w,\overline{z}-\overline{w})\mathcal{O}_k(w,\overline{w})\,. \tag{2.128}$$

the sum ranges over all operators in the theory and the *operator product coefficients* $C_{ij}{}^k$ are determined by the behaviour of the fields under fractional linear transformations. In $d=2$, the coefficients in the OPEs are of the form $C_{ij}{}^k = c_{ij}{}^k\,(z-w)^\alpha(\overline{z}-\overline{w})^\beta$ for some α, β and constants $c_{ij}{}^k$ [30].

Some operators are classed as *primary* depending on their OPE with the stress-energy tensor: if it is of the form [30]

$$T^X(z)\mathcal{O}(w,\overline{w}) \sim \frac{h}{(z-w)^2}\mathcal{O}(w,\overline{w}) + \frac{\partial_w\mathcal{O}(w,\overline{w})}{z-w} + \ldots\,, \tag{2.129}$$

$$\overline{T}^X(\overline{z})\mathcal{O}(w,\overline{w}) \sim \frac{h}{(\overline{z}-\overline{w})^2}\mathcal{O}(w,\overline{w}) + \frac{\overline{\partial}_w\mathcal{O}(w,\overline{w})}{\overline{z}-\overline{w}} + \ldots\,, \tag{2.130}$$

(where the ellipses "…" indicate non-singular terms which don't contribute to contour integrals) then $\mathcal{O}$ is called a primary operator of weight $(h,\overline{h})$. $X^\mu(z,\overline{z})$ is not a well-behaved conformal field because of the logarithmic behaviour of its two-point function,

Recall that we saw in Eq. (2.12) that Weyl invariance implied that the stress-energy tensor is traceless, $T^{X\alpha}{}_\alpha = 0$; this is the defining feature of a conformal field theory. It may hold at the classical level but fail due to quantum corrections as in, for example, Yang-Mills theory. We will see that tracelessness *does* fail for the X^μ CFT in general but in the critical dimension it cancels the contribution from the ghost CFT.

Any holomorphic field of weight h (*i.e.* one that transforms as in Eq. (2.46) with $n_+ - n_- = h$ can be Laurent expanded as [30]

$$A(z) = \sum_{n\in\mathbf{Z}-h} A_n z^{-n-h}\,, \qquad\qquad A_n = \frac{1}{2\pi\mathrm{i}}\oint\frac{\mathrm{d}z}{z^{1-n-h}}\,A(z)\,, \tag{2.131}$$

a similar statement holds for anti-holomorphic fields of weight $\overline{h}$ being expressed as Laurentz series in $\overline{z}$.

We know that X^μ is the sum of a holomorphic left-moving and anti-holomorphic right-moving part (Eq. (2.39)), from which it follows that ∂X^μ_L and $\overline{\partial}X^\mu_R$ are holomorphic and anti-holomorphic, respectively, and therefore they admit Laurent expansions in z and $\overline{z}$, respectively. Since X^μ is a worldsheet scalar, $\partial_\alpha X^\mu$ has weight $h=1$. We can write

$$\partial X^\mu_L(z) = -\mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbf{Z}}\alpha^\mu_n\,z^{-n-1}\,; \qquad \overline{\partial}X^\mu_R(\overline{z}) = -\mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbf{Z}}\tilde{\alpha}^\mu_n\,\overline{z}^{-n-1}\,. \tag{2.132}$$

We can write down expressions for α^μ_n and $\tilde{\alpha}^\mu_n$ using contour integrals around 0: we get [30]

$$\alpha^\mu_n = \frac{\mathrm{i}}{\sqrt{2\alpha'}}\oint\frac{\mathrm{d}z}{2\pi\mathrm{i}}\partial X^\mu_L(z)\,z^n\,; \qquad \tilde{\alpha}^\mu_n = \frac{\mathrm{i}}{\sqrt{2\alpha'}}\oint\frac{\mathrm{d}\overline{z}}{2\pi\mathrm{i}}\overline{\partial}X^\mu_R(\overline{z})\,\overline{z}^n\,. \tag{2.133}$$

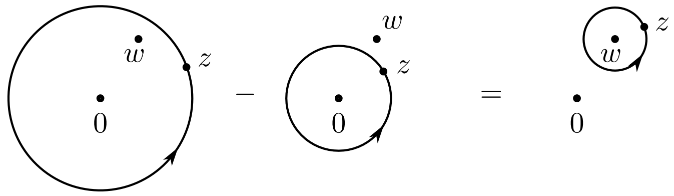


Figure 2.5: Radial-ordered commutators with (anti-) analytic fields expressed as a contour integral.

These have the commutation relations

$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu} \,. \tag{2.134}$$

T^X and $\overline{T}^X$ are actually not examples of primary fields themselves; for a generic CFT, T^X satisfies [30]

$$T^X(z)T^X(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T^X(w)}{(z-w)^2} + \frac{\partial_w T^X(w)}{z-w} + \ldots \tag{2.135}$$

and there is an analogous OPE for $\overline{T}^X$ with itself. In the quantum theory, T^X no longer transforms as a tensor but as a 'projective connection' with [20]

$$T'^X(z')(\mathrm{d}z')^2 = T^X(z)(\mathrm{d}z)^2 - \frac{c}{12}\{z';z\}(\mathrm{d}z)^2 \,, \tag{2.136}$$

where $\{\ ;\ \}$ is the *Schwarzian derivative* which can be expressed in several equivalent ways [66, 41, 67]

$$\{f;z\} \equiv \frac{f'''}{f'} - \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2 \equiv \Big(\frac{f''}{f'}\Big)' - \frac{1}{2}\Big(\frac{f''}{f'}\Big)^2 \equiv -2\sqrt{f'}\Big(\frac{1}{\sqrt{f'}}\Big)'' \,, \tag{2.137}$$

where all the primes indicate derivatives with respect to z. $\{,\}$ satisfies the following chain rule: $\{gf,z\} = \{g,f(z)\}(f'(z))^2 + \{f,z\}$, and vanishes for Möbius maps $f(z) = \frac{az+b}{cz+d}$ as is clear from the last formula in Eq. (2.137) since $1/\sqrt{f'(z)} \propto cz+d$.

We saw in section 2.1.2 that conformal transformations generate conserved currents (Eq. (2.43)); it follows from Stokes' theorem that there is a radially conserved charge $Q_{(\epsilon)}$ given by a contour integral [30]:

$$Q_{(\epsilon)} = \frac{1}{\alpha'}\oint \frac{\mathrm{d}z}{2\pi\mathrm{i}}\epsilon(z)T^X(z) + \frac{1}{\alpha'}\oint \frac{\mathrm{d}\overline{z}}{2\pi\mathrm{i}}\overline{\epsilon}(\overline{z})\overline{T}^X(\overline{z}) \,, \tag{2.138}$$

where the contours depend on where $Q_{(\epsilon)}$ appears in the radially-ordered correlation function (*i.e.* they should have all of the insertion points of operators to the right-hand-side of $Q_{(\epsilon)}$ on their *inside* and all of the insertion points of operators to the left-hand-side of $Q_{(\epsilon)}$ on their *outside*).

The commutator of an analytic (or anti-analytic) field like $T^X(z)$ with another field $\phi(w,\overline{w})$ (inside a radial-ordered correlation function) can be written in terms of a contour integral around w: see Fig. 2.5 [30], we have

$$[Q_{(\epsilon)}, \phi(w, \overline{w})] = \frac{1}{\alpha'} \oint_w \frac{\mathrm{d}z}{2\pi \mathrm{i}} \epsilon(z) T^X(z) \phi(w, \overline{w}) + \frac{1}{\alpha'} \oint_{\overline{w}} \frac{\mathrm{d}\overline{z}}{2\pi \mathrm{i}} \overline{\epsilon}(\overline{z}) \overline{T}^X(\overline{z}) \phi(w, \overline{w}) \,. \tag{2.139}$$

$T(z)$ and $\overline{T}(\overline{z})$ are fields of weight $(2,0)$ and $(0,2)$ so they can be Laurent expanded as in Eq. (2.131) as [30]

$$T^X(z) = \sum_{n \in \mathbf{Z}} L_n^X z^{-n-2} \,; \qquad L_n^X = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} T^X(z) z^{n+1} \tag{2.140}$$

and similarly $\overline{T}(\overline{z})$ can be expanded in terms of $\overline{L}_n$. Evaluating the radially-ordered commutator as in Eq. (2.139), we can use the OPE for $T(z)$ (Eq. (2.135)) to get

$$[L_m^X, L_n^X] = \frac{c}{12}(m^3 - m)\delta_{m+n,0} + (m-n) L_{m+n}^X \,; \tag{2.141}$$

the $\overline{L}_m^X$'s satisfy the same algebra and commute with the L_m's. This algebra is called the *Virasoro algebra*. Similarly it can be shown that any analytic field $\phi(z)$ which can be Laurent expanded in A_n (as in Eq. (2.131)) satisfies

$$[L_m^X, A_n] = ((h-1)m - n) A_{m+n} \,. \tag{2.142}$$

2.2.4 BRST Quantization

Theories with a gauge symmetry, *i.e.* theories whose mathematical description is redundant in that there are many equivalent ways to describe the same physical data, actually admit a deeper symmetry which was found by Becchi, Rouet and Stora [68] and Tyutin [69]. The analysis was first applied to the quantization of strings in [70] and [71].

Suppose we have a quantum field theory invariant under some local symmetry. Call the fields ϕ_i where i is a general label; it can distinguish between different types of fields. In our case both X^μ and $g_{\alpha\beta}$ are together the ϕ_i. There is a symmetry group which we parameterize with the infinitesimal transformations δ_α which satisfy an algebra

$$[\delta_\alpha, \delta_\beta] = f^\gamma{}_{\alpha\beta} \delta_\gamma \tag{2.143}$$

(this is like how the infinitesimal elements of a Lie group—equivalently the tangent space at the identity—automatically form a Lie algebra with the commutator bracket but in our case the space of symmetry parameters is infinite-dimensional). A general group element is denoted by the linear combination $\epsilon^\alpha \delta_\alpha$. Note that the index α here is an abstract index representing the spacetime indices $\mu, \ldots$, the worldsheet indices $\alpha, \beta, \ldots$ and the worldsheet coordinate σ all at the same time. An expression with Einstein summation requires integration over the worldsheet coordinates variables.

We gauge fix using a functional F^A, where again the index A can depend on the the worldsheet coordinate, to impose a gauge condition $F^A(\phi) = 0$. For example, in the lightcone gauge this imposes constraints on X^+, X^- and $g_{\alpha\beta}$; in conformal gauge all constraints are imposed on $g_{\alpha\beta}$.

We can impose the constraint by inserting a δ functional into the path integral:

$$\delta(F^A(\phi)) = \int \mathcal{D}B_A \exp(\mathrm{i}B_A F^A) \tag{2.144}$$

and then the path integral becomes

$$\int \mathcal{D}\phi_i \exp(-S)\frac{1}{\mathrm{Vol}} \to \int \mathcal{D}\phi_i \mathcal{D}B_A \mathcal{D}B_A \mathcal{D}c^\alpha \mathcal{D}b_A \exp(-S + \mathrm{i}B_A F^A - b_A c^\alpha \delta_\alpha F^A) \tag{2.145}$$

where the factor

$$\int \mathcal{D}b_A \mathcal{D}c^\alpha \exp(-b_A c^\alpha \delta_\alpha F^A) \tag{2.146}$$

is the general expression for a Faddeev-Popov determinant. We can think of this as a single action $S' = S + S_{\text{gauge}} + S_{\text{ghost}}$ where we have defined a gauge-fixing action $S_{\text{gauge}} = -\mathrm{i}B_A F^A$ and a ghost action $S_{\text{ghost}} = b_A c^\alpha \delta_\alpha F^A$. The new action S' is integrated over the original fields ϕ_i, the ghost fields b_A and c^α, and a new field 'conjugate' to F^A, B_A. S' has a an additional symmetry, Becchi-Rouet-Stora-Tyutin (BRST) symmetry; it is invariant under the BRST transformation δ_B whose action on the fields is given by

$$\begin{aligned} \delta_B \phi_i &= -\mathrm{i}\epsilon c^\alpha \delta_\alpha \phi_i && (2.147)\\ \delta_B B_A &= 0 && (2.148)\\ \delta_B b_A &= \epsilon B_A && (2.149)\\ \delta_B c^\alpha &= \frac{\mathrm{i}}{2}\epsilon f^\alpha{}_{\beta\gamma} c^\beta c^\gamma. && (2.150) \end{aligned}$$

Now because the ϕ_i and B_A are taken to be commuting whereas b_A and c^α are anti-commuting, it is necessary for ϵ to be anti-commuting so that these transformations preserve the commutation type of the fields.

Note that the original action S is invariant by itself under this action because it is just a gauge transformation paramaterized by ϵc^α, and the action is gauge invariant.

The variation in the other two terms cancels out; this follows from the anti-commuting properties of c^α and the Lie algebra axioms the structure constants satisfy. Therefore the BRST variation is nilpotent,

$$\delta_B(\delta_{\mathrm{B}}) = 0. \tag{2.151}$$

The other important property of the transformation is that [72]

$$\delta_B(b_A F^A) = \mathrm{i}\epsilon(S_{\text{gauge}} + S_{\text{ghost}})\,. \tag{2.152}$$

Suppose we make an infinitesimal change in the gauge-fixing functional F^A, $F^A \mapsto F^A + \epsilon\delta F^A$. S does not depend on F so it does not change, and S' changes as

$$S' \mapsto S' + \epsilon\delta(S_{\text{ghost}} + S_{\text{gauge}}) = -\mathrm{i}\delta_B(b_A \delta F^A) \tag{2.153}$$

where we have used Eq. (2.152). Then the change in the matrix element between an initial and final state $\langle f|i\rangle$ can be shown with a path integral manipulation to be given by

$$\epsilon\delta\langle f|i\rangle = \mathrm{i}\langle f|\delta_B(b_A\delta F^A)|i\rangle. \tag{2.154}$$

We define a conserved BRST charge Q_B whose anti-commutator gives the BRST transformation

$$\mathrm{i}\epsilon\{Q_\mathrm{B}, Y\} = \delta_B(Y) \tag{2.155}$$

so we have

$$\delta\langle f|i\rangle = -\langle f|\epsilon\{Q_\mathrm{B}, b_A\delta F^A\}|i\rangle. \tag{2.156}$$

We want amplitudes to be stationary under a small variation of the action so we stipulate that physical states will satisfy

$$\langle\psi'|\epsilon\{Q_\mathrm{B}, b_A\delta F^A\}|\psi\rangle = 0. \tag{2.157}$$

We need this to hold for arbitrary variations δF^A; this means it is necessary to enforce

$$Q_\mathrm{B}|\psi\rangle = 0 \tag{2.158}$$

for any physical state $|\psi\rangle$, i.e. we enforce that physical states must be BRST invariant, or *closed*. Physical states defined in this way form a subspace of the Hilbert space $\mathscr{H}$ called $\mathscr{H}_\mathrm{closed}$.

If we change our gauge choice F^A then the functional dependence of the hamiltonian changes, but we want to be able to alter our gauge choice while Q_B remains conserved. In particular we want Q_B to commute with the variation in the action

$$[Q_\mathrm{B}, \{Q_\mathrm{B}, b_A\delta F^A\}] = [Q_\mathrm{B}^2, b_A\delta F^A] = 0 \tag{2.159}$$

and this holds for arbitrary δF^A only if $Q_\mathrm{B}^2 = 0$. We can't have $Q_\mathrm{B}^2 =$ constant because Q_B has ghost number 2.

We can check that the action of δ_B on the fields given above is indeed nilpotent using the anti-commutation property of the c^α and the Lie algebra axioms satisfies by $f^\alpha{}_{\beta\gamma}$.

The nilpotence of Q_B means that from an arbitrary state $|\chi\rangle$ we can get a physical state $Q_\mathrm{B}|\chi\rangle$, which is automatically annihilated by Q_B. We call a state of the form $Q_\mathrm{B}|\chi\rangle$ *exact*. Exact states form a subspace of the Hilbert space called $\mathscr{H}_\mathrm{exact}$.

The hermiticity of Q_B can be used to see that an exact state is annihilated by any physical state (including itself or any exact state) $|\psi\rangle$ since

$$\langle\psi|Q_\mathrm{B}|\chi\rangle = \underbrace{(Q_\mathrm{B}|\psi\rangle)^\dagger}_{=0}|\chi\rangle = 0. \tag{2.160}$$

Then if $|\psi\rangle$ is any physical state, $|\psi'\rangle = |\psi\rangle + Q_\mathrm{B}|\chi\rangle$ for any state $|\chi\rangle$ is also physical, and moreover it has the same inner product with any other physical state $|\zeta\rangle$ as $|\psi\rangle$ does:

$$\langle\zeta|\psi'\rangle = \langle\zeta|\psi\rangle + \underbrace{\langle\zeta|Q_{\rm B}|\chi\rangle}_{=0} = \langle\zeta|\psi\rangle. \tag{2.161}$$

Then the two states $|\psi\rangle$ and $|\psi'\rangle$ are physically equivalent, and we can define an equivalence relation $\sim_{\rm B}$ by $|\psi_1\rangle \sim_{\rm B} |\psi_2\rangle$ whenever $|\psi_2\rangle = |\psi_1\rangle + Q_{\rm B}|\chi\rangle$ for some $|\chi\rangle$.

The space of physically distinct states is then

$$\mathscr{H}_{\rm BRST} = \mathscr{H} / \sim_{\rm B} = \frac{\mathscr{H}_{\rm closed}}{\mathscr{H}_{\rm exact}}. \tag{2.162}$$

When we have a nilpotent operator such as $Q_{\rm B}$, the image of the operator is a subspace of its kernel, and the space of closed forms quotiented out by the exact forms is called a cohomology group. Then the space of physical and physically distinct states is a cohomology group.

2.2.5 BRST quantization of the string

As discussed in section 2.2.2, the gauge symmetry of the bosonic string can be fixed by the introduction of a Faddeev-Popov determinant, expressed in terms of a (b, c)-ghost system on the worldsheet.

As we did for the matter fields in Eq. (2.8), we can define a stress tensor for the ghost fields in terms of the functional derivative of the ghost action with respect to the worldsheet metric, obtaining

$$T^{\rm gh}_{\alpha\beta} = -\frac{4\pi\alpha'}{\sqrt{-h}}\frac{\delta S^{\rm gh}}{h^{\alpha\beta}}\,; \tag{2.163}$$

we get a symmetric traceless tensor given in the conformal gauge as [17]

$$T^{\rm gh} = \alpha'(c\,\partial b + 2(\partial c)b)\,, \tag{2.164}$$

with a similar equation for $\overline{T}^{\rm gh}$ in the case of the closed string. Imposing the equal time anti-commutation relations in light-cone co-ordinates $\{b_{++}(\sigma,\tau), c^{+}(\sigma',\tau)\} = 2\pi\delta(\sigma-\sigma')$, it can be found that in complex coordinates the radial quantization of the (b, c) system can be expressed as

$$c(z) = \sum_{n\in\mathbf{Z}} \frac{c_n}{z^{n-1}}\,; \qquad b(z) = \sum_{n\in\mathbf{Z}} \frac{b_n}{z^{n+2}}\,; \qquad \{c_m, b_n\} = \delta_{m+n,0}\,, \tag{2.165}$$

with all other (anti-)commutators vanishing. As we did for the matter stress-energy tensor T^X in Eq. (2.140), we can expand $T^{\rm gh}(z)$ in modes as

$$T^{\rm gh}(z) = \sum_{n\in\mathbf{Z}} L^{\rm gh}_n z^{-n-2}\,; \qquad L^{\rm gh}_n = \oint \frac{{\rm d}z}{2\pi{\rm i}} T^{\rm gh}(z) z^{n+1}\,. \tag{2.166}$$

The mode expansion of $L^{\rm gh}$ is

$$L_n^{\rm gh} = \sum_{n\in\mathbf{Z}} (2m-n) b_{m+n} c_{-n} \,, \tag{2.167}$$

which satisfy the algebra

$$[L_m^{\rm gh}, L_n^{\rm gh}] = (m-n) L_{m+n}^{\rm gh} + \frac{1}{6}(m - 13m^3)\delta_{m+n,0} \,. \tag{2.168}$$

When $L_m^{\rm gh}$ and L_m^X are combined to give a single operator L_m defined as

$$L_m = L_m^X + L_m^{\rm gh} - \delta_{m,0} \,, \tag{2.169}$$

the L_m's satisfy the algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3 - m) + \frac{1}{6}(m - 13m^3) + 2m \,. \tag{2.170}$$

In the case $D = 26$, the central term vanishes. The structure constants of the symmetry algebra therefore have a simple form and we can construct the BRST charge Q_B for bosonic string theory. Whenever we have an action with a gauge symmetry generated by elements of a Lie algebra spanned by K_i with structure constants $[K_i, K_j] = f_{ij}{}^k K_k$, with ghosts b_i and c^i which transform in the adjoint and dual-adjoint representation, respectively, we can construct a nilpotent operator [17]

$$Q = c^i K_i - \frac{1}{2} f_{ij}{}^k c^i c^j b_k \,. \tag{2.171}$$

The Virasoro algebra generated by the L_m's is a suitable candidate for this Lie algebra when $D = 26$, with the b_n and c_n modes transforming appropriately. Then we can construct the following BRST operator for the open bosonic string:

$$\begin{aligned}
Q &= \sum_{m\in\mathbf{Z}} L_{-m}^X c_m - \frac{1}{2}\sum_{m,n\in\mathbf{Z}} (m-n) : c_{-m} c_{-n} b_{m+n} : - c_0 && (2.172)\\
&= \sum_{n\in\mathbf{Z}} : (L_{-n}^X + \frac{1}{2} L_{-n}^{\rm gh} - \delta_{n,0}) c_n : && (2.173)\\
&= \oint \frac{{\rm d}z}{2\pi{\rm i}} : c\Big(T^X + \frac{1}{2} T^{\rm gh}\Big) : && (2.174)\\
&= \oint \frac{{\rm d}z}{2\pi{\rm i}} : c\Big(-\frac{1}{4\alpha'}\partial X \cdot \partial X + (\partial c) b\Big) : \,. && (2.175)
\end{aligned}$$

The space of physical states can be calculated from this, see *e.g.* section 4.3 of [62]. States are written in terms of the SL$(2,\mathbf{R})$-invariant state, $|0;0\rangle$, which is annihilated by α_n^μ for $n \geq 0$, by c_n for $n \geq 2$ and by b_n for $n \geq -1$. Then the states

$$|k\rangle \equiv : {\rm e}^{{\rm i}k\cdot X} : c_1 |0;0\rangle \tag{2.176}$$

with $k^2 = -\frac{1}{\alpha'}$ are Q-closed; these correspond to the tachyon of the theory. At the next level, the massless excitations which are the ones we are interested in from the point of view of studying QFT in the $\alpha' \to 0$ limit, before computing the BRST cohomology there is a 28-dimensional vector space of states spanned by

$$|S_1^\mu\rangle = \mathcal{N}\alpha_{-1}^\mu|k\rangle\,; \qquad |S_2\rangle = c_{-1}|k\rangle\,; \qquad |S_3\rangle = b_{-1}|k\rangle\,; \tag{2.177}$$

where $\mu = 0,\dots,25$ and now $k^2 = 0$. Let us find the cohomology of Q. To begin, we compute $Q|S_1^\mu\rangle$. Using the expression for Q in Eq. (2.175), we can begin with the contribution coming from the T^X term. Using the mode expansion of T^X, this term becomes

$$\frac{\mathcal{N}}{8}\sum_{\ell,n}\oint\frac{\mathrm{d}z}{2\pi\mathrm{i}}:\frac{c(z)}{z^{\ell+2}}\alpha_{\ell-n}\cdot\alpha_n:\alpha_{-1}^\mu|k\rangle \tag{2.178}$$

From the commutation relations Eq. (2.134) we can calculate $[\alpha_{\ell-n}\cdot\alpha_n,\alpha_{-1}^\mu] = (\delta_{n1} + \delta_{\ell-n,1})\alpha_{\ell-1}^\mu$ which we can insert in Eq. (2.178) yielding

$$\frac{\mathcal{N}}{8}\sum_{\ell}\oint\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{c(z)}{z^{\ell+2}}2\alpha_{\ell-1}^\mu|k\rangle + \frac{\mathcal{N}}{8}\sum_{\ell,n}\oint\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{c(z)}{z^{\ell+2}}\alpha_{-1}^\mu:\alpha_{\ell-n}\cdot\alpha_n:|k\rangle\,. \tag{2.179}$$

Let us calculate the first term in Eq. (2.179). From the mode expansion of the string coordinates X^μ (*e.g.* Eq. (2.7.26) of [62]):

$$X^\mu(z,\overline{z}) = x^\mu - \mathrm{i}\alpha' p^\mu\log|z|^2 + \mathrm{i}\Big(\frac{\alpha'}{2}\Big)^{\frac{1}{2}}\sum_{m\neq 0}\frac{\alpha_m^\mu}{m}(z^{-m}+\overline{z}^{-m})\,, \tag{2.180}$$

we can calculate

$$[\alpha_m^\mu, :\mathrm{e}^{\mathrm{i}k\cdot X(w)}:] = \Big(\frac{\alpha'}{2}\Big)^{\frac{1}{2}}(w^m+\overline{w}^m)k^\mu:\mathrm{e}^{\mathrm{i}k\cdot X(w)}:\,. \tag{2.181}$$

Using this, the first term of Eq. (2.179) becomes:

$$\frac{\mathcal{N}}{4}:\mathrm{e}^{\mathrm{i}k\cdot X}:\sum_{\ell}c_{-\ell}c_1\Big(\Big(\frac{\alpha'}{2}\Big)^{\frac{1}{2}}(w^{\ell-1}+\overline{w}^{\ell-1})k^\mu + \alpha_{\ell-1}^\mu\Big)\Big|_{w=0}|0;0\rangle\,. \tag{2.182}$$

For $\ell \geq 2$, the summand vanishes because $\alpha_{\ell-1}^\mu$ annihilates the vacuum or because $w^{\ell-1}\to 0$. For $\ell = 1$, the oscillator mode $\alpha_{\ell-1}^\mu \propto p^\mu$ annihilates the vacuum but the other term doesn't vanish; indeed it is equal to

$$\frac{\mathcal{N}}{2}\Big(\frac{\alpha'}{2}\Big)^{\frac{1}{2}}k^\mu|S_2\rangle\,. \tag{2.183}$$

For $\ell \leq -2$, $c_{-\ell}$ annihilates $|0;0\rangle$ and for $\ell = -1$, $c_{-\ell}c_1 = 0$. The case $\ell = 0$ remains; starting from Eq. (2.179) it is easy to check that it is given by $\frac{1}{4}c_0|S_1^\mu\rangle$ and we will see that it cancels a contribution from the T^{gh} term of $Q|S_1^\mu\rangle$.

The second term in Eq. (2.179) can be calculated with the use of

$$: \alpha_{\ell-n} \cdot \alpha_n : |k\rangle = c_1 \Big([: \alpha_{\ell-n} \cdot \alpha_n :, : \mathrm{e}^{\mathrm{i}k\cdot X(w)} :] + : \mathrm{e}^{\mathrm{i}k\cdot X(w)} :: \alpha_{\ell-n} \cdot \alpha_n : \Big)\Big|_{w=0} |0\rangle \quad (2.184)$$

The second term on the right hand side of Eq. (2.184), when inserted in the second term of Eq. (2.179), gives

$$\frac{\mathcal{N}}{8} \sum_{\ell,n} c_{-\ell} \alpha^\mu_{-1} c_1 : \mathrm{e}^{\mathrm{i}k\cdot X} :: \alpha_{\ell-n} \cdot \alpha_n : |0;0\rangle = 0 \quad (2.185)$$

which vanishes because at least one of $c_{-\ell}$, $\alpha^\mu_{\ell-n}$, α^μ_n must be an annihilation operator.

To evaluate the commutator in Eq. (2.184), we use Eq. (2.181) to get

$$\begin{aligned} &[\alpha_{\ell-n} \cdot \alpha_n, : \mathrm{e}^{\mathrm{i}k\cdot X(w)} :] \qquad (2.186) \\ &\quad = \Big(\frac{\alpha'}{2}\Big)^{\frac{1}{2}} : \mathrm{e}^{\mathrm{i}k\cdot X(w)} : \big((w^n + \overline{w}^n) \alpha_{\ell-n} \cdot k + (w^{\ell-n} + \overline{w}^{\ell-n}) \alpha_n \cdot k \big) . \end{aligned}$$

where use has been made of the fact that $\alpha_{\ell-n} \cdot k$ can be commuted past $: \mathrm{e}^{\mathrm{i}k\cdot X(w)} :$ since all possible commutators are proportional to $k^2 = 0$.

Now, when either $\ell - n \geq 1$ or $n \geq 1$, Eq. (2.186) annihilates the vacuum, because in each term there is either a positive power of w which vanishes in the limit $w \to 0$, or there is a positive-moded α_m which annihilates the vacuum. When $n = 0$, $\alpha^\mu_n \propto p^\mu$ annihilates the vacuum but $(w^n + \overline{w}^n)$ doesn't vanish, and similarly when $\ell - n = 0$ the first term annihilates the vacuum thanks to $\alpha^\mu_{\ell-n}$ but the second doesn't vanish, so Eq. (2.186) becomes

$$2\Big(\frac{\alpha'}{2}\Big)^{\frac{1}{2}} : \mathrm{e}^{\mathrm{i}k\cdot X(w)} : \big(\delta_{n0} + \delta_{n\ell}\big) \alpha_\ell \cdot k . \quad (2.187)$$

Inserting this into Eq. (2.184) and expanding $c(z)$ in modes, we see that the second term of Eq. (2.179) becomes

$$\frac{\mathcal{N}}{2} \Big(\frac{\alpha'}{2}\Big)^{\frac{1}{2}} \sum_{\ell \leq -1} c_{-\ell} \alpha^\mu_{-1} c_1 : \mathrm{e}^{\mathrm{i}k\cdot X} : \alpha_\ell \cdot k |0;0\rangle \quad (2.188)$$

$|0;0\rangle$ is annihilated by $k \cdot \alpha_\ell$ for $\ell \geq 0$ and by $c_{-\ell}$ for $\ell \leq -2$, while for $\ell = -1$ the expression vanishes due to the two factors of c_1. Therefore since both Eq. (2.185) and Eq. (2.189) vanish, so does the second term in Eq. (2.179).

The T^{gh} term in $Q|S^\mu_1\rangle$ is equal to

$$\begin{aligned} &\oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} : c(\partial c) b : \Big|_z \mathcal{N} \alpha^\mu_{-1} |k\rangle \qquad (2.189) \\ &\quad = \mathcal{N} \alpha^\mu_{-1} : \mathrm{e}^{\mathrm{i}k\cdot X} : \Big|_0 \sum_{\ell,m,n} (1-m) \delta_{\ell+m+n,0} : c_\ell c_m b_n : c_1 |0\rangle . \end{aligned}$$

Since $\ell + m + n = 1$, at least one of the operators c_ℓ, c_m, b_n is an annihilation operator or vanishes due to nilpotence, but in the case $n = -1$, b_n doesn't anti-commute past c_1 and we need to use $b_{-1}c_1|0;0\rangle = |0;0\rangle$. Then the $n=1$ term in Eq. (2.189) is

$$\mathcal{N}\alpha^\mu_{-1} :\mathrm{e}^{\mathrm{i}k\cdot X}: \Big|_0 \sum_\ell \ell : c_\ell c_{(1-\ell)} : |0\rangle = -c_0|S^\mu_1\rangle \tag{2.190}$$

which cancels the $\ell=0$ term from Eq. (2.179). Therefore $Q|S^\mu_1\rangle$ is given completely by Eq. (2.183).

Next we can find the action of Q on $|S_2\rangle$. The first term coming from the "T^X" part of Eq. (2.175), *i.e.*

$$\oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} c(z)\Big(-\frac{1}{4\alpha'}\partial X\cdot\partial X\Big)\Big|_z c_{-1}c_1 :\mathrm{e}^{\mathrm{i}k\cdot X}: \Big|_0 |0;0\rangle \tag{2.191}$$

can be calculated using the OPE of $\partial X^\mu(z)$ with $:\mathrm{e}^{\mathrm{i}k\cdot X(w)}:$. To get a normal-ordered expression, we must commute the positive frequency operators $\alpha^\mu_{m\geq 0}$ past $:\mathrm{e}^{\mathrm{i}k\cdot X(w)}:$, using the mode expansion for $\partial X^\mu(z)$ in Eq. (2.132) and the commutator in Eq. (2.181), giving us a geometric series in w/z and $\overline{w}/z$ which can be evaluated at $w=0$ yielding

$$\partial X^\mu(z) :\mathrm{e}^{\mathrm{i}k\cdot X(0)}: = -\frac{\mathrm{i}}{z}\alpha' k^\mu :\mathrm{e}^{\mathrm{i}k\cdot X(0)}: + :\partial X^\mu(z)\mathrm{e}^{\mathrm{i}k\cdot X(0)}:, \tag{2.192}$$

which gives us the OPE

$$\partial X^\mu(z) :\mathrm{e}^{\mathrm{i}k\cdot X(0)}: \sim -\frac{\mathrm{i}}{z}\alpha' k^\mu :\mathrm{e}^{\mathrm{i}k\cdot X(0)}: . \tag{2.193}$$

Inserting this in Eq. (2.191) and then finding the residue of the contour integral after expanding $c(z)$ and the second $\partial X^\mu(z)$ in modes, we see that Eq. (2.191) is equal to

$$\frac{1}{2}\left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} k_\mu \sum_{n\in\mathbf{Z}} c_n\alpha_{-n}c_{-1}c_1 :\mathrm{e}^{\mathrm{i}k\cdot X}: \Big|_0 |0;0\rangle . \tag{2.194}$$

Now, c_n annihilates the vacuum for $n\geq 2$ while a_{-n} annihilates the vacuum for $n\leq 0$ (as before, there are terms coming from commuting it past $:\mathrm{e}^{\mathrm{i}k\cdot X}:$ but these are all proportional to $k^2=0$). The only remaining contribution comes from $n=1$, but this term also vanishes because it contains two copies of c_1, *i.e.* , the "T^X part" of $Q|S_2\rangle$ is zero.

To compute the second term in $Q|S_2\rangle$ coming from T^{gh}, we expand $b(z)$ in modes and find

$$\oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} c(\partial c) b\Big|_z c_{-1}c_1 :\mathrm{e}^{\mathrm{i}k\cdot X}: \Big|_0 |0;0\rangle . \tag{2.195}$$

It can be seen that this expression vanishes unless $n=1$ or $n=-1$; the contribution from these two terms sums to give The only non-zero terms come from $n=-1$ and $n=1$ which sum to give

$$\frac{1}{2}\partial^2(c\partial c)c :\mathrm{e}^{\mathrm{i}k\cdot X}: \Big|_0 |0;0\rangle - \frac{1}{2}c(\partial c)\partial^2 c :\mathrm{e}^{\mathrm{i}k\cdot X}: \Big|_0 |0;0\rangle , \tag{2.196}$$

but this also vanishes because $\partial^2(c\partial c)c = c(\partial c)(\partial^2 c)$, and therefore the "$T^{\mathrm{gh}}$ part" and hence all of $Q|S_2\rangle$ is equal to zero.

Lastly, we should calculate $Q|S_3\rangle$. Beginning with the contribution from the "T^{gh} part" of Q, we expand the fields in modes, obtaining

$$\sum_{\ell,m,n}(1-m)\delta_{\ell+m+n,0}\;:c_\ell c_m b_n:\;\mathrm{e}^{\mathrm{i}k\cdot X}|0;0\rangle \tag{2.197}$$

where we've used $b_{-1}c_1|0;0\rangle = |0;0\rangle$. As above, at least one of the operators is an annihilation operator because of the Kronecker δ, so all terms in the sum vanish and the contribution from this term is zero.

To calculate the other term coming from the "T^X part" of Q, we use the OPE of $\partial X^\mu(z)$ with $:\mathrm{e}^{\mathrm{i}k\cdot X(w)}:$ in Eq. (2.193). Using this OPE to replace $\partial X^\mu(z):\mathrm{e}^{\mathrm{i}k\cdot X(0)}:$, we find

$$\begin{aligned}\frac{1}{2}\Big(\frac{\alpha'}{2}\Big)^{\frac{1}{2}}k_\mu\sum_{m,n\in\mathbf{Z}}c_m\alpha^\mu_n\oint\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{1}{z^{m+n+1}}:\mathrm{e}^{\mathrm{i}k\cdot X}:\Big|_0|0;0\rangle&\\ =\frac{1}{2}\Big(\frac{\alpha'}{2}\Big)^{\frac{1}{2}}k_\mu\sum_{n\in\mathbf{Z}}c_{-n}\alpha^\mu_n:\mathrm{e}^{\mathrm{i}k\cdot X}:\Big|_0|0;0\rangle&\end{aligned} \tag{2.198}$$

Now, $|0;0\rangle$ is annihilated for all values of $n\neq -1$ ($k\cdot\alpha_n$ doesn't commute with $:\mathrm{e}^{\mathrm{i}k\cdot X}:$ but as before, the commutators are all proportional to $k^2=0$). $Q|S_3\rangle$ is therefore equal to the $n=-1$ term in Eq. (2.198), and we have

$$Q|S_3\rangle = \frac{1}{\mathcal{N}}\frac{1}{2}\Big(\frac{\alpha'}{2}\Big)^{\frac{1}{2}}k_\mu|S^\mu_1\rangle\,. \tag{2.199}$$

To summarize, the action of Q on the massless sector of the bosonic string is given by

$$Q\alpha^\mu_{-1}|k\rangle\propto k^\mu c_{-1}|k\rangle\,;\qquad Qb_{-1}|k\rangle\propto k\cdot\alpha_{-1}|k\rangle\,;\qquad Qc_{-1}|k\rangle=0\,, \tag{2.200}$$

From this, it follows that only the 26-dimensional subspace spanned by $c_{-1}|k\rangle$ and $\epsilon\cdot\alpha_{-1}|k\rangle$ where $k\cdot\epsilon=0$ is Q-closed. Moreover, the two dimensional space of states spanned by $c_{-1}|k\rangle$ and by $k\cdot\alpha_{-1}|k\rangle$ is Q-exact, and therefore the Q-cohomology of massless physical states is 24-dimensional.

It is interesting to note that (as pointed out in section 4.3 of [62]) the action of Q on the massless sector of the bosonic string is isomorphic to the action of the BRST variation δ_B on the field content of Yang-Mills gauge theory, at least at the non-interacting level. If A^a_μ is a gauge field and c^a and $\bar{c}^a$ are the ghost and anti-ghost field, respectively, then we have

$$\delta_B\big(A^a_\mu\big)\propto\partial_\mu c^a\,;\qquad \delta_B\big(\bar{c}^a\big)\propto\partial\cdot A^a\,;\qquad \delta_B\big(c^a\big)=0\,, \tag{2.201}$$

plus non-linear terms. There is a clear isomorphism between Eq. (2.201) and Eq. (2.200). The significance of this that it motivates us to guess that the $\alpha'\to 0$ limit of open string theory matches Yang-Mills theory not only at the level of summed-up amplitudes, but rather that we can expect to isolate the contributions coming from the various sectors of the worldsheet theory and find that they individually match the corresponding terms coming from Feynman diagrams on the Yang-Mills side. We will see later that the correspondence seems to be valid even in the interacting theory, because we can find a diagram-by-diagram matching between the two theories even at the two-loop level.

The BRST quantization of the RNS string proceeds along analogous lines to the bosonic string, except the construction is more complicated.

2.3 The Schottky group

The first attempts at writing down multiloop amplitudes for the bosonic string were made in the very early days of dual resonance models [73, 5, 6, 74, 75]. These amplitudes were constructed by sewing together multi-reggeon (*i.e.* open string) vertices [76, 77, 78] which were found, in turn, by factorizing the Veneziano amplitude [23]. It was quickly noticed that the string diagrams were integrals over the moduli space of Riemann surfaces, where the Riemann surfaces were naturally defined using Schottky groups.

The basic idea of the Schottky group is that h-loop Riemann surfaces are represented as the quotient of the Riemann sphere or the upper-half-plane (with a discrete set of points removed) by h Möbius maps. Each Möbius map can be specified by three parameters; these constitute the moduli which are integrated over.

We can also use super-projective transformations to sew handles onto the super-Riemann sphere $\mathbf{CP}^{1|1}$ by quotienting; these SRS's correspond to worldsheets for superstring amplitudes in which NS states are propagating along the sewn handles (because quotienting by a super-projective transformation is equivalent to sewing two NS punctures). Conversely, worldsheets in which R states propagate around handles must be formed by sewing pairs of R punctures, which correspond to a different type of singularity in the superconformal structure of the surface (see *e.g.* section 4 of [60] and [79]).

2.3.1 Projective transformations

Before describing super-projective transformations and super-Schottky groups which are the appropriate tools for super-Riemann surfaces, we recall the main points about the analogous quantities for Riemann surfaces.

As we have seen in Eq. (2.51) and Eq. (2.52), a projective transformation maps the Riemann sphere $\mathbf{CP}^1$ to itself, and can be represented in homogeneous coordinates by a 2×2 matrix or in a local complex coordinate by a fractional linear transformation.

Let us use homogeneous coordinates (z_u, z_d), with $z \equiv z_u/z_d$ when $z_d \neq 0$. We are be interested in projective transformations with two distinct eigenvectors $(u_u, u_d)^{\mathrm{t}}$ and $(v_u, v_d)^{\mathrm{t}}$, called fixed points[2], and an eigenvalue $\sqrt{k}$ satisfying $|\sqrt{k}| < 1$, where k is called the multiplier (since $\det S = 1$, the other eigenvalue must then equal $1/\sqrt{k}$). The action

[2]For the sake of simplicity, when $z_d \neq 0$, we can choose the representative with $z_d = 1$, but one should keep in mind that the bra and ket introduced here are projective objects, which can appear only in relations that are unchanged when they are rescaled.

of these transformations can be described by the following bracket notation for the points of the Riemann surface

$$|z\rangle = \begin{pmatrix} z_u \\ z_d \end{pmatrix}, \qquad \langle z| \equiv \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} z_u \\ z_d \end{pmatrix} \right]^{\mathrm{t}} \equiv [I|z\rangle]^{\mathrm{t}} = (z_d, -z_u) . \tag{2.202}$$

The eigenvector associated to the eigenvalue $\sqrt{k}$ is called the *attractive* fixed point, and the other one is called the *repulsive* fixed point. To see why this is so, note that an arbitrary point on $\mathbf{CP}^1$ can be written as a sum of the two eigenvectors, and then $S^n|z\rangle$ converges to one of the two fixed points as $n \to \pm\infty$

$$|z\rangle = \lambda_1|u\rangle + \lambda_2|v\rangle \qquad \Rightarrow \qquad S^n|z\rangle = \lambda_1\Big(\frac{1}{\sqrt{k}}\Big)^n|u\rangle + \lambda_2(\sqrt{k})^n|v\rangle \tag{2.203}$$

which converges, in a projective sense, to $|u\rangle$ for $n \to \infty$ and to $|v\rangle$ for $n \to -\infty$. With this definition of the bra-vector we can follow the notation of [60], and introduce a skew-symmetric bilinear form $\langle w|z\rangle$ which is proportional to the difference between the coordinates of the two points. Indeed, $\langle w|z\rangle \equiv z_u w_d - z_d w_u = -\langle z|w\rangle$. Therefore, if $z_d, w_d \neq 0$, $\langle w|z\rangle = z_d w_d\,(z - w)$. In this language, we can write a projective transformation S in terms of its multiplier k, and of the fixed-point kets $|u\rangle$ and $|v\rangle$, as

$$\begin{aligned} S &= 1\!\!1 + \frac{1}{\langle v|u\rangle}\left[\Big(-k^{\frac{1}{2}}+1\Big)|v\rangle\langle u| - \Big(-k^{-\frac{1}{2}}+1\Big)|u\rangle\langle v|\right] \\ &= k^{-\frac{1}{2}}\left(1\!\!1 + \frac{1-k}{\langle v|u\rangle}|v\rangle\langle u|\right), \end{aligned} \tag{2.204}$$

where the second form is obtained by using $1\!\!1 = (|u\rangle\langle v| - |v\rangle\langle u|)/\langle v|u\rangle$. The sign of the square root of k is immaterial, since both choices define the same projective transformation (the situation will be different in the supersymmetric case). It is easy to verify that S turns into S^{-1} under the exchange $|u\rangle \leftrightarrow |v\rangle$ and that the bra corresponding to the ket $|Sz\rangle = S|z\rangle$ is simply $\langle Sz| = \langle z|S^{-1}$, so that the bilinear form is invariant under projective transformations: $\langle Sz|Sw\rangle = \langle z|w\rangle$. A single bracket, however, is not a well-defined object, as it depends on the representative chosen for z and w; as is well-known, one can form the first projective invariant by using four points, since in this case all z_d components cancel in the ratio

$$(z_1, z_2, z_3, z_4) = \frac{\langle z_1|z_2\rangle\langle z_3|z_4\rangle}{\langle z_3|z_2\rangle\langle z_1|z_4\rangle} = \frac{(z_2 - z_1)(z_4 - z_3)}{(z_2 - z_3)(z_4 - z_1)} . \tag{2.205}$$

We want to ensure S has one eigenvalue satisfying $|\sqrt{k}| < 1$ so we wish to exclude the case when the two eigenvalues, say $\lambda_\pm$, both have absolute value 1. If $|\lambda_+|^2 = \lambda_+\overline{\lambda}_+ = 1$ then since $\lambda_+\lambda_- = 1$, in the case we wish to exclude we have $\overline{\lambda}_+ = \lambda_-$. We can compute the real and imaginary components

$$\begin{aligned} \Re(\lambda_+) &= \frac{1}{2}(\lambda_+ + \overline{\lambda}_+) = \frac{1}{2}(\lambda_+ + \lambda_-) = \frac{1}{2}\mathrm{Tr}(S) \\ \Im(\lambda_+) &= \frac{1}{2}(\lambda_+ - \overline{\lambda}_+) = \frac{1}{2}(\lambda_+ - \lambda_-) = \frac{1}{2}\sqrt{\mathrm{Tr}(S)^2 - 4}. \end{aligned}$$

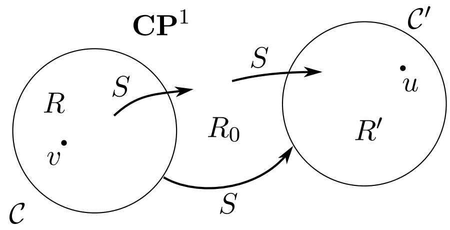


Figure 2.6: Given a Möbius map and a circle separating the two fixed points, we can partition $\mathbf{CP}^1$ in a natural way into a fundamental domain for S and two circular discs containing the fixed points.

Then since $\sqrt{\mathrm{Tr}(S)^2 - 4}$ is pure imaginary we will have $\mathrm{Tr}(S)^2 \le 4$ in the excluded case. Since $\mathrm{Tr}(S)$ is real we will also have $\mathrm{Tr}(S)^2 \ge 0$ in this excluded case.

Hence, we can ensure that there is always an eigenvalue with $|\lambda_i| < 1$ by requiring

$$\mathrm{Tr}(S)^2 \notin [0, 4]. \tag{2.206}$$

Möbius maps satisfying this property are called *loxodromic*. If the eigenvalues are real then is it *hyperbolic*.

The projective transformations we have introduced have the property that they map circles to circles (counting lines in the complex plane as circles through the point at infinity on the Riemann sphere) [80]. We can see this because any Möbius transformation can be written as a composition of translation $z \mapsto z + b$, multiplication $z \mapsto az$, and inversion in the unit circle, $z \mapsto -1/z$. Explicitly, we can decompose a general Möbius transformation as inversion conjugated by two affine-linear maps:

$$z \mapsto \frac{az+b}{cz+d} = \left(z \mapsto \left(\frac{a}{c} - \frac{b}{d}\right) z + \frac{a}{c}\right) \circ \left(z \mapsto -\frac{1}{z}\right) \circ \left(z \mapsto \frac{c}{d} z + 1\right). \tag{2.207}$$

It is obvious that translation and dilatation preserve circles. Inversion in the unit circle preserves circles because we can show with algebraic manipulation that if[3]

$$R^2 = \|z - C\|^2 = \|z\|^2 (1 - \bar{C}/z - C/\bar{z}) + \|C\|^2 \tag{2.208}$$

then

$$\frac{R^2}{(R^2 - \|C\|^2)^2} = \frac{1}{\|z\|^2} - \frac{\bar{C}/\bar{z} + C/z}{R^2 - \|C\|^2} + \frac{\|C\|^2}{(R^2 - \|C\|^2)^2} = \left\| -\frac{1}{z} - \frac{\bar{C}}{R^2 - \|C\|^2} \right\|^2, \tag{2.209}$$

so the map $z \mapsto -1/z$ takes a circle with centre C and radius R to a circle with centre $\frac{\bar{C}}{R^2 - \|C\|^2}$ and radius $R^2/(R^2 - \|C\|^2)^2$.

A Möbius map taking any circle $\mathcal{C}$ to any other circle $\mathcal{C}'$ can be constructed by picking three points on $\mathcal{C}$ and three points on $\mathcal{C}'$, and using the fact that for any two triples z_1, z_2, z_3; w_1, w_2, w_3 of points in $\mathbf{CP}^1$ there is a Möbius map S with $S(z_i) = w_i$, and that

[3] Here $\|\cdot\|$ denotes the absolute value of a complex number: $\|x + \mathrm{i}\, y\| = \sqrt{x^2 + y^2}$ for $x, y \in \mathbf{R}$.

a circle is completely determined by any three distinct points on it.

Consider a circle $\mathcal{C}$ on $\mathbf{CP}^1$ separating the attractive fixed point u from the repulsive fixed point v of some projective transformation S. Then the circle $\mathcal{C}'$ defined as the image of $\mathcal{C}$ under S separates u from $\mathcal{C}$. With these two circles we can partition $\mathbf{CP}^1$ into three pieces: the region containing u bounded by $\mathcal{C}$ (say R); the region containing v bounded by $\mathcal{C}'$ (say R'), and the remainder (say R_0) (see Fig. 2.6). This has the property that S maps the inside of $\mathcal{C}$ (*i.e.* R) to the outside of $\mathcal{C}'$ (*i.e.* $R \cup R_0$) and the outside of $\mathcal{C}$ (*i.e.* $R_0 \cup R'$) to the inside of $\mathcal{C}'$ (*i.e.* R'). From this it follows that R_0 (plus one of the two circles, $\mathcal{C}$ or $\mathcal{C}'$) is a fundamental domain for the action of S on $\mathbf{CP}^1 - \{u, v\}$, which is to say that it contains exactly one representative of each equivalency class of the relation $z \sim S(z)$. In particular, for $S(z) = \frac{az+b}{cz+d}$ with $c \neq 0$ we can choose a pair of such circles $\mathcal{C}$ and $\mathcal{C}'$ in the complex plane such that $S(\mathcal{C}) = \mathcal{C}'$ and S is an isometry at $\mathcal{C}$ with the Euclidean metric. The circles are centered respectively in a/c and $-d/c$, and both have radius $1/|c|$. The quotient space $(\mathbf{CP}^1 - \{u, v\})/S$ is a Riemann surface; it can be obtained from $\mathbf{CP}^1$ schematically by cutting out R and R' and gluing together the two circles $\mathcal{C}$ and $\mathcal{C}'$ so it is topologically a torus. We can always make a change of coordinates such that the two fixed points are at $u = 0$ and $v = \infty$, then S has the form $z \mapsto S(z) = kz$, so k is the sole modulus for the torus.[4]

It is not actually necessary for the boundaries of the fundamental region to be circles; they may as well be any Jordan curves with the same topology so long as one is the image of the other under S.

Note that this is equivalent to 'sewing' a handle to the surface at the two points u and v. To sew two points P_1 and P_2 on a Riemann surface means to take a pair of complex coordinates charts z_i which vanish at the points, *i.e.* $z_i(P_i) = 0$, and then to remove the points P_i and identify the points in rest of the two charts via $z_1 z_2 \equiv -k$. k is called the sewing parameter. We've already seen how a change of coordinates mapping $u \mapsto 0$ and $v \mapsto \infty$ can put any Möbius map into the form $z \sim kz$. z is the coordinate chart which vanishes at u, and we can take $w = -1/z$ to be the coordinate chart which vanishes at v. Then the equivalence relation we are imposing takes the form $zw \sim -k$, *i.e.* we are sewing u to v.

Higher-loop Riemann surfaces (*i.e.* those with multiple handles or boundaries) can be constructed in a similar way. For a compact Riemann surface with h handles, we take h Möbius maps S_μ, $\mu = 1, \ldots, h$ such that we can find on the Riemann sphere $2h$ circles $\mathcal{C}_\mu$, $\mathcal{C}_{\mu'}$ satisfying $S_\mu(\mathcal{C}_\mu) = \mathcal{C}_{\mu'}$, with $\mathcal{C}_\mu$ around the repulsive fixed point v_μ and $\mathcal{C}_{\mu'}$ around the attractive fixed point u_μ.[5] Let R_μ denote the region inside $\mathcal{C}_\mu$, let $R_{\mu'}$ denote the region inside $\mathcal{C}_{\mu'}$, and let R_0 denote the region outside all $2h$ circles.

The Schottky group $\mathcal{S}$ is the free group generated by the S_μ's, *i.e.* it is the group of all (reduced) words which can be written as a sequence of S_μ's and their inverses. Before quotienting $\mathbf{C}$ by $\mathcal{S}$, we need to remove the limit set Λ, which is the set of accumulation

[4] The torus is usually parametrized by a modulus τ, where the torus is constructed as $\mathbf{C}/\sim$ where $w \sim w+1 \sim w+\tau$. This coordinate w is related to z by $z = \mathrm{e}^{2\pi \mathrm{i} w}$, so the moduli are related via $k = \mathrm{e}^{2\pi \mathrm{i}\tau}$.

[5] On $\mathbf{CP}^1$ there is no meaning of the inside and outside of a circle; so we need just that these statements hold in some chart.

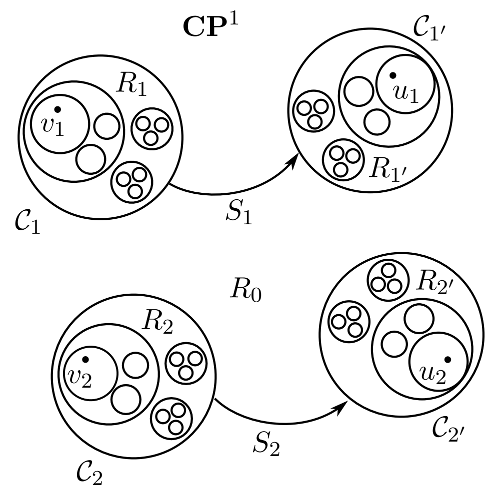


Figure 2.7: Schottky circles for a genus 2 Riemann surface.

points of the orbits of $\mathcal{S}$. To characterize Λ, we can consider an infinite set of nested circles obtained by acting on the $2h$ defining circles with the elements of $\mathcal{S}$. There is a correspondence between circles and Schottky group words: there are $2h$ circles at the first level (corresponding to the h Schottky generators and their inverses), and each circle at the nth level has $(2h-1)$ of the $(n+1)$th-level circles inside it, corresponding to the fact that there are $(2h-1)$ ways to get a length-$(n+1)$ word from a length-n word by multiplying on the left (since multiplying by the inverse of the left-most element gives a length-$(n-1)$ word). We can define the precise correspondence between circles and Schottky words recursively by saying that if α and β are two Schottky words, then the circle $\mathcal{C}_{\alpha\cdot\beta}$ is defined as the image $T_\alpha(\mathcal{C}_\beta)$, and by identifying $\mathcal{C}_\alpha = \mathcal{C}_\mu$ for $T_\alpha = S_\mu^{-1}$ and $\mathcal{C}_\alpha = \mathcal{C}_{\mu'}$ for $T_\alpha = S_\mu$. With this definition, we see that if α is a length-n reduced word, then if z is a point in R_0, then $T_\alpha(z)$ lies inside the level-n circle C_α but outside all of the level-$(n+1)$ circles. There are $2h(2h-1)^{n-1}$ circles at level n. A point is in the limit set Λ if and only if it is inside of a circle at level n for every $n \in \mathbf{N}$. There are uncountably many limit points; there is a 1–1 correspondence between limit points and infinite Schottky words.

In Fig. 2.7 the first three levels of Schottky circles are shown for a genus $h = 2$ Riemann surface constructed with a Schottky group generated by two Möbius maps S_1 and S_2. Each level-n circle contains $(2h-1) = 3$ level-$(n+1)$ circles [81].

Once the limit set Λ has been substracted, we may quotient by the action of $\mathcal{S}$ and we will obtain a genus-g Riemann surface. Topologically, quotienting by $\mathcal{S}$ is equivalent to cutting out the insides of each of the $2h$ generating circles and gluing them pairwise along their boundaries, so each pair of circles gives one handle on the quotient surface. Conventionally, we take the a_μ-cycles on the quotient surface to be homologous to the generating circles $\mathcal{C}_\mu$, while the b_μ-cycles go along the handles we have just added such that b_μ connects a point z to $S_\mu(z)$ on the covering surface $\mathbf{CP}^1 - \Lambda$. Fig. 2.8 illustrates this for $h = 2$ handles. Note that constructing Riemann surfaces with Schottky groups puts the a_μ and b_μ cycles on different footings, so the behaviour of various formulae

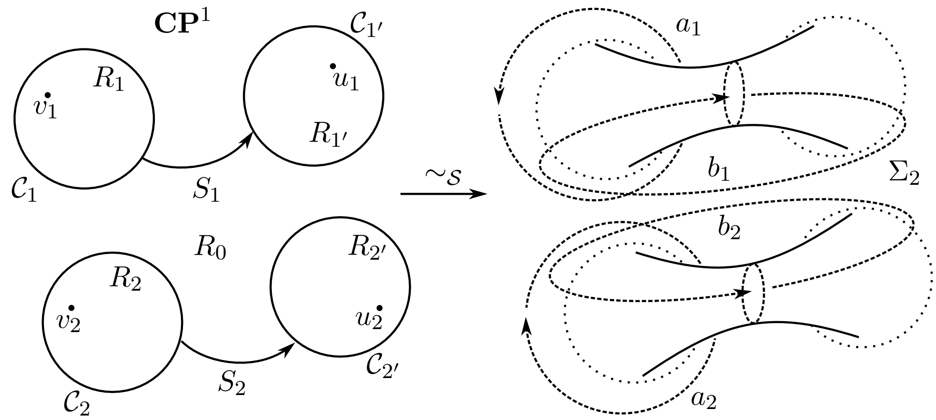


Figure 2.8: Quotienting by the Schottky group to glue two handles onto a Riemann surface.

under modular transformations mixing a_μ and b_μ cycles is often obscured. Other ways of constructing higher-genus Riemann surfaces, such as quotienting the upper-half plane by a Fuchsian group [41], do not have this drawback. We can see the origin of the dimension of moduli space in this construction: the quotient surface Σ_h is completely determined by the Schottky group $\mathcal{S}_h$ which in turn is completely specified by listing h generating Möbius maps. Each of these can be specified by giving three complex numbers, *i.e.* the two fixed points u_μ, v_μ and the multiplier k_μ, so we have $3h$ complex parameters. In fact, some of these are redundant (as coordinates on moduli space) because we can always use a global automorphism to fix any three points, for example, to fix $u_1 = 0$, $v_1 = \infty$ and $v_2 = 1$; the rest of the parameters are the moduli and for closed string worldsheets we find $\dim_{\mathbf{C}}(\mathcal{M}_h) = 3h - 3$ as expected.

We are interested in open string worldsheets, so the Riemann surfaces we use are bordered Riemann surfaces (recall that this means Riemann surfaces which are locally biholomorphic with $\overline{\mathbf{H}}$, the upper-half plane). These can be constructed with Schottky groups by starting with $\overline{\mathbf{H}}$ instead of $\mathbf{CP}^1$, and then quotienting by a particular Schottky group. Note that the Schottky group will have to map the border of $\overline{\mathbf{H}}$ to itself, *i.e.* , it will have to fix the extended real line. This implies that the fixed points and the multipliers are all real, or equivalently, so are the $\mathrm{PSL}(2, \mathbf{C})$ matrices, *i.e.* this type of Schottky group is a subgroup of $\mathrm{PSL}(2, \mathbf{R})$. The same moduli-counting argument goes through as before, except that the moduli are all constrained to be real so $3h - 3$ is the *real* dimension of moduli space, *i.e.* it is half as big for the open string case as for the closed string case with the same number of loops. Fig. 2.9 illustrates how an open string worldsheet can be constructed by adding two boundaries to $\overline{\mathbf{H}}$ with a Schottky subgroup of $\mathrm{PSL}(2, \mathbf{R})$.

As discussed in section 2.1.5, it is usually useful to formulate superstring theory on *super*-Riemann surfaces; the additional structure they have requires the Schottky group to be suitably modified.

2.3.2 Super-projective transformations

Super-projective transformations are automorphisms of the super-Riemann sphere $\mathbf{CP}^{1|1}$, which is defined in terms of homogeneous coordinates in $\mathbf{C}^{2|1}$ by the equivalence relation

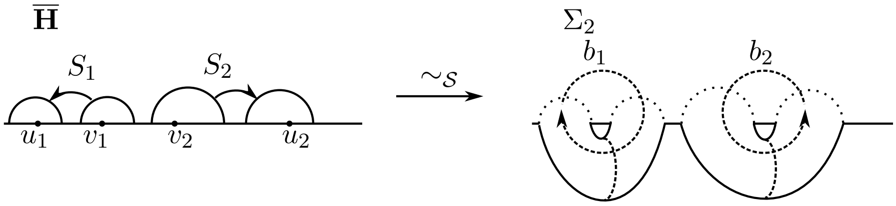


Figure 2.9: Constructing an open string worldsheet with a Schottky subgroup of $\mathrm{PSL}(2, \mathbf{R})$.

$(z_1, z_2|\theta) \sim (\lambda z_1, \lambda z_2|\lambda\theta)$ for non-zero complex λ, where the bosonic coordinates z_1 and z_2 are not allowed to vanish simultaneously. To fix the superconformal structure, we may specify a holomorphic 1-form on $\mathbf{CP}^{1|1}$ that is homogenous of degree 2 in $z_1, z_2|\zeta$; such a form is [60]

$$\varpi = z_1 \mathrm{d}z_2 - z_2 \mathrm{d}z_1 - \theta \mathrm{d}\theta. \tag{2.210}$$

If we use scaling symmetry to set $z_2 = 1$ and write $z_1 = z$, then $\varpi = \mathrm{d}z - \theta\mathrm{d}\theta$ which is orthogonal to $\mathcal{D}_\theta = \partial_\theta + \theta\partial_z$, so $z|\theta$ are superconformal coordinates for $z_1 \neq 0$. If we define a skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the homogeneous co-ordinates by

$$\langle z, y \rangle = z_1 y_2 - z_2 y_1 - \zeta\psi \tag{2.211}$$

for $y = (y_1, y_2|\psi)$, then we may write $\varpi = \langle z, \mathrm{d}z \rangle$. A linear map on $\mathbf{C}^{2|1}$ will therefore preserve ϖ, and hence the superconformal structure, if it preserves $\langle \cdot, \cdot \rangle$. The group of such transformations is $\mathrm{OSp}(1|2)$, which can be realised by matrices of the form

$$\mathbf{S} = \left(\begin{array}{cc|c} a & b & \alpha \\ c & d & \beta \\ \hline \gamma & \delta & e \end{array}\right) \tag{2.212}$$

where the 5 bosonic and 4 fermionic variables are subject to the 2 fermionic and 2 bosonic constraints,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -\delta \\ \gamma \end{pmatrix} \qquad ad - bc - \alpha\beta = 1 \qquad e = 1 - \alpha\beta \tag{2.213}$$

so the group has dimension $3|2$. We can find an $\mathrm{OSp}(1|2)$ matrix taking $\mathbf{u} = (u_1, u_2|\theta)$ and $\mathbf{v} = (v_1, v_2|\phi)$ to points equivalent to $(0, 1|0)$ and $(1, 0|0)$ respectively; one such matrix is

$$\mathbf{\Gamma_{uv}} = \frac{1}{\sqrt{\langle \mathbf{u}, \mathbf{v} \rangle}} \left(\begin{array}{cc|c} u_2 & -u_1 & \theta \\ v_2 & -v_1 & \phi \\ \hline \frac{u_2\phi - v_2\theta}{\sqrt{\langle \mathbf{u}, \mathbf{v} \rangle}} & \frac{v_1\theta - u_1\phi}{\sqrt{\langle \mathbf{u}, \mathbf{v} \rangle}} & \sqrt{\langle \mathbf{u}, \mathbf{v} \rangle} - \frac{\theta\phi}{\sqrt{\langle \mathbf{u}, \mathbf{v} \rangle}} \end{array}\right). \tag{2.214}$$

We have one bosonic degree of freedom remaining; we can stipulate that a point $\mathbf{w} = (w_1, w_2|\omega)$ is mapped to a point equivalent to $(1, 1|\Theta_{\mathbf{uwv}})$ where there is no freedom in choosing the fermionic co-ordinate, which is therefore a super-projective invariant of $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$. The image of $\mathbf{w}$ under $\mathbf{\Gamma_{uv}}$ is

$$\mathbf{\Gamma_{uv}w} = \frac{1}{\sqrt{\langle \mathbf{u}, \mathbf{v}\rangle}}\Big(\langle \mathbf{w}, \mathbf{u}\rangle, \langle \mathbf{w}, \mathbf{v}\rangle\Big|\frac{\theta\langle \mathbf{v}, \mathbf{w}\rangle + \phi\langle \mathbf{w}, \mathbf{u}\rangle + \omega\langle \mathbf{u}, \mathbf{v}\rangle + \omega\theta\phi}{\sqrt{\langle \mathbf{u}, \mathbf{v}\rangle}}\Big). \tag{2.215}$$

A general dilatation of the superconformal co-ordinates corresponds to the OSp(1|2) matrix

$$\mathbf{P}(y) = \left(\begin{array}{cc|c} y^{\frac{1}{2}} & 0 & 0 \\ 0 & y^{-\frac{1}{2}} & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \tag{2.216}$$

which leaves invariant the points $(0, 1|0)$ and $(1, 0|0)$. We may use a transformation like this to scale the bosonic coordinates of $\mathbf{\Gamma_{uv}w}$ as desired, obtaining

$$\mathbf{P}\Big(\frac{\langle \mathbf{w}, \mathbf{v}\rangle}{\langle \mathbf{w}, \mathbf{u}\rangle}\Big)\mathbf{\Gamma_{uv}w} \sim \Big(1, 1\Big|\frac{\theta\langle \mathbf{w}, \mathbf{v}\rangle + \omega\langle \mathbf{v}, \mathbf{u}\rangle + \phi\langle \mathbf{u}, \mathbf{w}\rangle + \theta\omega\phi}{\sqrt{\langle \mathbf{v}, \mathbf{u}\rangle\langle \mathbf{u}, \mathbf{w}\rangle\langle \mathbf{w}, \mathbf{v}\rangle}}\Big) \tag{2.217}$$

giving us an explicit expression for the odd super-projective invariant $\Theta_{\mathbf{z_1z_2z_3}}$:

$$\Theta_{\mathbf{z_1z_2z_3}} = \frac{\zeta_1\langle \mathbf{z}_2, \mathbf{z}_3\rangle + \zeta_2\langle \mathbf{z}_3, \mathbf{z}_1\rangle + \zeta_3\langle \mathbf{z}_1, \mathbf{z}_2\rangle + \zeta_1\,\zeta_2\,\zeta_3}{\sqrt{\langle \mathbf{z}_1, \mathbf{z}_2\rangle\langle \mathbf{z}_2, \mathbf{z}_3\rangle\langle \mathbf{z}_3, \mathbf{z}_1\rangle}}, \tag{2.218}$$

where $\mathbf{z}_i = z_i|\zeta_i$, as in Eq. (3.222) of [39].

As with projective transformations, super-projective transformations preserve cross-ratios of the form

$$\widehat{\Psi}_{\mathbf{z}_1\mathbf{z}_2\mathbf{z}_3\mathbf{z}_4} = \frac{\langle \mathbf{z}_1, \mathbf{z}_2\rangle}{\langle \mathbf{z}_1, \mathbf{z}_4\rangle}\frac{\langle \mathbf{z}_3, \mathbf{z}_4\rangle}{\langle \mathbf{z}_3, \mathbf{z}_2\rangle}, \tag{2.219}$$

but there is a novelty in the super-projective case. In the non-supersymmetric case, any cross-ratio of four points can be expressed simply in terms of any other cross-ratio of the same four points, but the analogous statement does not hold. Instead, we need to include the fermionic invariants, getting identities like

$$\widehat{\Psi}_{\mathbf{z}_1\mathbf{z}_2\mathbf{z}_3\mathbf{z}_4} + \widehat{\Psi}_{\mathbf{z}_1\mathbf{z}_3\mathbf{z}_2\mathbf{z}_4} - (\widehat{\Psi}_{\mathbf{z}_1\mathbf{z}_3\mathbf{z}_2\mathbf{z}_4})^{\frac{1}{2}}\Theta_{\mathbf{z}_1\mathbf{z}_3\mathbf{z}_2}\Theta_{\mathbf{z}_1\mathbf{z}_4\mathbf{z}_2} = 1\,. \tag{2.220}$$

This can be checked quickly by noting that the left-hand side is OSp(1|2)-invariant and fixing 3|2 convenient superconformal co-ordinates e.g. $\mathbf{z_1} = 0|0$, $\mathbf{z}_2 = \infty|0$, $\mathbf{z}_3 = \eta|\theta$, $\mathbf{z}_4 = 1|\phi$, in which case it becomes simply

$$(1 - \eta + \theta\phi) + \eta - \sqrt{\eta}\frac{\theta}{\sqrt{\eta}}\phi = 1\,. \tag{2.221}$$

Note that for $|k| < 1$, $\mathbf{P}(k)$ has $0|0$ as an attractive fixed point and $\infty|0$ as a repulsive fixed point. Using $\mathbf{\Gamma_{uv}}$ to map a pair of points $\mathbf{u}$ and $\mathbf{v}$ to $0|0$ and $\infty|0$ respectively, it follows that

$$\mathbf{S} = \boldsymbol{\Gamma}_{\mathbf{uv}}^{-1}\mathbf{P}(k)\boldsymbol{\Gamma}_{\mathbf{uv}} \tag{2.222}$$

has $\mathbf{u}$ as an attractive fixed point and $\mathbf{v}$ as a repulsive fixed point. Our skew-symmetric quadratic form $\langle\cdot,\cdot\rangle$ can be expressed in terms of a bra-ket notation $\langle\mathbf{u}|\mathbf{v}\rangle = -\langle\mathbf{u},\mathbf{v}\rangle$ where

$$\langle\mathbf{u}| = (u_2, -u_1, \theta) \qquad\qquad |\mathbf{u}\rangle = (u_1, u_2|\theta)^t \tag{2.223}$$

satisfying $\langle\mathbf{u}|\mathbf{v}\rangle = -\langle\mathbf{u},\mathbf{v}\rangle$. It is related to the super-difference between two points $\mathbf{z} \dot{-} \mathbf{w}$: if $|\mathbf{z}\rangle = (z\lambda_1, \lambda_1, \psi\lambda_1)^t$ and $|\mathbf{w}\rangle = (w\lambda_2, \lambda_2, \omega\lambda_2)^t$ for λ_1, $\lambda_2 \neq 0$, then

$$\langle\mathbf{w}|\mathbf{z}\rangle = \lambda_1\lambda_2(\mathbf{z} \dot{-} \mathbf{w}) = \lambda_1\lambda_2(z - w - \psi\omega)\,. \tag{2.224}$$

For cross-ratios, the λ_i all cancel so we can use either notation:

$$\frac{\langle\mathbf{z}_1|\mathbf{z}_2\rangle}{\langle\mathbf{z}_1|\mathbf{z}_4\rangle}\frac{\langle\mathbf{z}_3|\mathbf{z}_4\rangle}{\langle\mathbf{z}_3|\mathbf{z}_2\rangle} = \frac{\mathbf{z}_1 \dot{-} \mathbf{z}_2}{\mathbf{z}_1 \dot{-} \mathbf{z}_4}\frac{\mathbf{z}_3 \dot{-} \mathbf{z}_4}{\mathbf{z}_3 \dot{-} \mathbf{z}_2}\,. \tag{2.225}$$

The bra-ket notation has the benefit of allowing us to write $\mathbf{S}$ defined in Eq. (2.222) as

$$\mathbf{S} = \mathbb{1} + \frac{1}{\langle\mathbf{v}|\mathbf{u}\rangle}\left[\left(-k^{\frac{1}{2}}+1\right)|\mathbf{v}\rangle\langle\mathbf{u}| - \left(-k^{-\frac{1}{2}}+1\right)|\mathbf{u}\rangle\langle\mathbf{v}|\right]\,. \tag{2.226}$$

$\mathbf{S}$ defined this way satisfies

$$\frac{\langle\mathbf{S}(\mathbf{z}),\mathbf{u}\rangle}{\langle\mathbf{S}(\mathbf{z}),\mathbf{v}\rangle} = k\frac{\langle\mathbf{z},\mathbf{u}\rangle}{\langle\mathbf{z},\mathbf{v}\rangle}\,. \tag{2.227}$$

2.3.3 Super Schottky groups

Quotienting $\mathbf{CP}^{1|1}$ by the action of $\mathbf{S}$ defined in Eq. (2.226) is equivalent to putting a pair of NS punctures at $\mathbf{u}$ and $\mathbf{v}$ and sewing them with a sewing parameter proportional to k. Topologically, this has the same effect (at least on the reduced space $\mathbf{CP}^1$) as cutting out discs around $\mathbf{u}$ and $\mathbf{v}$ and identifying their boundaries, so this quotient adds a handle to the surface, increasing the genus by 1. The choice of spin structure along the handle is determined by the branch of $k^{\frac{1}{2}}$.

To build a genus-h SRS, we may repeat this sewing procedure h times, choosing h pairs of attractive and repulsive fixed points $\mathbf{u}_\mu = u_\mu|\theta_\mu$, $\mathbf{v}_\mu = v_\mu|\phi_\mu$ and g multipliers k_μ for $\mu = 1,\ldots,h$. The super-Schottky group $\overline{\mathcal{S}}_h$ is the group freely generated by

$$\mathbf{S}_\mu = \boldsymbol{\Gamma}_{\mathbf{u}_\mu\mathbf{v}_\mu}^{-1}\mathbf{P}(k_\mu)\boldsymbol{\Gamma}_{\mathbf{u}_\mu\mathbf{v}_\mu} \qquad\qquad \mu = 1,\ldots,h. \tag{2.228}$$

To obtain a genus h super-Riemann surface $\mathcal{M}_h$, we subtract the limit set Λ (*i.e.* the set of accumulation points of the orbits of $\overline{\mathcal{S}}_h$) from $\mathbf{CP}^{1|1}$ and then quotient by the action of the super-Schottky group:

$$\mathcal{M}_h = (\mathbf{CP}^{1|1} - \Lambda)/\mathbf{Sc}_h. \tag{2.229}$$

Note that the fixed points must be sufficiently far from each other and the multipliers sufficiently small that there exists a fundamental domain with the desired topology, i.e. that of $\mathbf{CP}^{1|1}$ with $2h$ discs cut out. The fixed points $\mathbf{u}_\mu$, $\mathbf{v}_\mu$ and the multipliers k_μ are moduli for the surface, but for $h \geq 2$ we can use the OSp(1|2) symmetry to fix 3|2 of these: $\mathbf{u}_1 = 0|0$, $\mathbf{v}_1 = \infty|0$, $\mathbf{v}_2 = 1|\Theta_{\mathbf{u}_1\mathbf{v}_2\mathbf{v}_1}$ (where $\Theta_{\mathbf{u}_1\mathbf{v}_2\mathbf{v}_1}$ is still an unfixed modulus) so the super-moduli space $\widehat{\mathfrak{M}}_h$ has complex dimension $3h-3|2h-2$.

To build multi-loop open superstring worldsheets this way, we should start with the super-disc $\mathbf{D}^{1|1}$ which can be obtained by quotienting $\mathbf{CP}^{1|1}$ by the involution $z|\theta \mapsto z^*|\theta^*$, so $\mathbf{RP}^{1|1}$ is its border. A super-projective map will be an automorphism of $\mathbf{D}^{1|1}$ if it preserves $\mathbf{RP}^{1|1}$, so we should build the super-Schottky group from super-projective transformations whose fixed points $\mathbf{u}_\mu$, $\mathbf{v}_\mu$ are in $\mathbf{R}^{1|1}$ and whose multipliers k_μ are real. If we quotient $\mathbf{D}^{1|1} - \Lambda$ by h of these, then we will get a SRS with $(h+1)$ borders (and no handles). The moduli space $\widehat{\mathfrak{M}}_h^{\text{open}}$ of such SRSs has *real* dimension $3h-3|2h-2$.

For the $h=2$ surfaces we are looking at, we use the OSp(1|2) symmetry to write the fixed points as in Eq. (4.24).

Multipliers

Every element $\mathbf{S}_\alpha$ of a super-Schottky group is similar to a matrix of the form $\mathbf{P}(k_\alpha)$ as in Eq. (2.216) for some k_α. We can find $k_\alpha{}^{\frac{1}{2}}$ using the cyclic property of the supertrace, obtaining a quadratic equation with roots

$$k_\alpha{}^{\frac{1}{2}} = \frac{\mathrm{sTr}(\mathbf{S}_\alpha) + 1 \pm \sqrt{(\mathrm{sTr}(\mathbf{S}_\alpha)+1)^2 - 4}}{2}\,, \tag{2.230}$$

one root being the inverse of the other. $k_\alpha{}^{\frac{1}{2}}$ is the one whose absolute value is less than 1.

We can expand the $k_\alpha{}^{\frac{1}{2}}$ as series in $k_i{}^{\frac{1}{2}}$. For $h=2$, using the fixed points Eq. (4.24) we find

$$\begin{aligned} k(\mathbf{S}_1\mathbf{S}_2)^{\frac{1}{2}} &= k_1{}^{\frac{1}{2}} k_2{}^{\frac{1}{2}} y + \mathcal{O}(k_\mu) && (2.231)\\ &= k_1{}^{\frac{1}{2}} k_2{}^{\frac{1}{2}}\, \widehat{\Psi}_{\mathbf{u}_1\mathbf{v}_1\mathbf{u}_2\mathbf{v}_2} + \mathcal{O}(k_\mu) && (2.232)\\ k(\mathbf{S}_1^{-1}\mathbf{S}_2)^{\frac{1}{2}} &= -k_1{}^{\frac{1}{2}} k_2{}^{\frac{1}{2}} \frac{y}{u} + \mathcal{O}(k_\mu) && (2.233)\\ &= k_1{}^{\frac{1}{2}} k_2{}^{\frac{1}{2}}\, \widehat{\Psi}_{\mathbf{v}_1\mathbf{u}_1\mathbf{u}_2\mathbf{v}_2} + \mathcal{O}(k_\mu)\,. && (2.234) \end{aligned}$$

where y is defined in Eq. (4.27). Note that $k(\mathbf{S}_1^{-1}\mathbf{S}_2)^{\frac{1}{2}}$ can be obtained from $k(\mathbf{S}_1\mathbf{S}_2)^{\frac{1}{2}}$ by swapping the attractive and repulsive fixed points of $\mathbf{S}_1$ in the cross-ratio.

Super period matrix

The super-abelian differentials are an h-dimensional space of holomorphic volume forms defined on a genus-h SRS. They are spanned by $\mathbf{\Omega}_\mu$, $\mu = 1, \ldots, h$ normalized by their integrals around the a-cycles:

$$\frac{1}{2\pi \mathrm{i}} \oint_{A_\mu} \mathbf{\Omega}_\nu = \delta_{\mu\nu}\,. \tag{2.235}$$

Their integrals around the b-cycles define the super-period matrix

$$\frac{1}{2\pi \mathrm{i}} \oint_{B_\mu} \boldsymbol{\Omega}_\nu = \boldsymbol{\tau}_{\mu\nu}. \tag{2.236}$$

The $\boldsymbol{\Omega}_\mu$ can be expressed in terms of the super-Schottky group as (equation (21) of [9])

$$\begin{aligned} \boldsymbol{\Omega}_\mu(z|\psi) &= [\mathrm{d}z|\mathrm{d}\psi] \sum_\alpha{}^{(\mu)} D_\psi \log \frac{\langle \mathbf{z}|\mathbf{T}_\alpha|\mathbf{u}_\mu\rangle}{\langle \mathbf{z}|\mathbf{T}_\alpha|\mathbf{v}_\mu\rangle} && (2.237) \\ &= [\mathrm{d}z|\mathrm{d}\psi] \sum_\alpha{}^{(\mu)} \Big[\frac{\langle \mathbf{z}|\Phi\,\mathbf{T}_\alpha|\mathbf{u}_\mu\rangle}{\langle \mathbf{z}|\mathbf{T}_\alpha|\mathbf{u}_\mu\rangle} - \frac{\langle \mathbf{z}|\Phi\,\mathbf{T}_\alpha|\mathbf{v}_\mu\rangle}{\langle \mathbf{z}|\mathbf{T}_\alpha|\mathbf{v}_\mu\rangle} \Big] && (2.238) \end{aligned}$$

where the sum $\sum_\alpha^{(\mu)}$ is over all elements of the super-Schottky group which don't have $\mathbf{S}_\mu^{\pm 1}$ as their right-most factor, D_ψ is the superconformal derivative $D_\psi = \partial_\psi + \psi \partial_z$, and Φ is the matrix

$$\Phi = \left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & -1 & 0 \end{array} \right). \tag{2.239}$$

Φ has the property that if $|\mathbf{z}\rangle = (\lambda z, \lambda|\lambda\psi)^t$ then

$$D_\psi \langle \mathbf{z}|\mathbf{w}\rangle = \langle \mathbf{z}|\Phi|\mathbf{w}\rangle \tag{2.240}$$

and that the map $z|\psi \mapsto \langle \mathbf{z}|\mathbf{w}\rangle | \langle \mathbf{z}|\Phi|\mathbf{w}\rangle$ is superconformal. The super period matrix can be computed as

$$\boldsymbol{\tau}_{\mu\nu} = \frac{1}{2\pi \mathrm{i}} \left[\delta_{\mu\nu} \log k_\mu - {}^{(\nu)}\sum_\alpha{}^{(\mu)} \log \frac{\langle \mathbf{u}_\nu|\mathbf{T}_\alpha|\mathbf{v}_\mu\rangle \langle \mathbf{v}_\nu|\mathbf{T}_\alpha|\mathbf{u}_\mu\rangle}{\langle \mathbf{u}_\nu|\mathbf{T}_\alpha|\mathbf{u}_\mu\rangle \langle \mathbf{v}_\nu|\mathbf{T}_\alpha|\mathbf{v}_\mu\rangle} \right]. \tag{2.241}$$

The sum is over all elements of the super-Schottky group which don't have $\mathbf{S}_\nu^{\pm 1}$ as their left-most element or $\mathbf{S}_\mu^{\pm 1}$ as their right-most element.

We can compute the leading terms in the small-k_μ expansion for $\boldsymbol{\tau}_{\mu\nu}$. For $h = 2$, using the fixed points in Eq. (4.24) we find

$$2\pi \mathrm{i}\, \boldsymbol{\tau} = \begin{pmatrix} \log k_1 - 2k_2^{\frac{1}{2}} \left(1 - \frac{1}{u}\right) \theta\phi & \log u \\ \log u & \log k_2 - 2k_1^{\frac{1}{2}} \left(1 - \frac{1}{u}\right) \theta\phi \end{pmatrix} + \mathcal{O}(k_\mu) \tag{2.242}$$

and so

$$\begin{aligned} 4\pi^2 \det(\mathrm{Im}\, \boldsymbol{\tau}) &= \log(k_1)\log(k_2) - \log(u)^2 \\ &\quad - 2(1 - 1/u)\big(\log(k_1) k_2^{\frac{1}{2}} + \log(k_2) k_1^{\frac{1}{2}}\big)\theta\phi + \mathcal{O}(k_\mu)\,. \end{aligned} \tag{2.243}$$

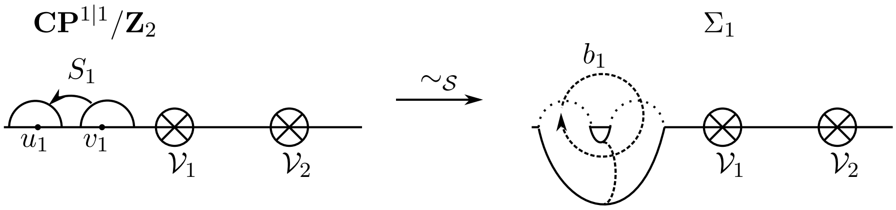


Figure 2.10: The construction of a one-loop two-point worldsheet with a super-Schottky group.

2.4 Example: two-point gluon amplitude at one loop

Let us consider an example super-string calculation in the RNS formalism in which we carry out an integration over super-moduli space with a potential ambiguity. Instead of considering a two-loop vacuum amplitude, we will consider a one-loop two point amplitude. The calculations are similar because both involve integration over the positions of four NS punctures positioned on the boundary of the super-upper-half plane $\mathbf{CP}^{1|1}/(z|\phi \sim z^*|\phi)$, characterized by one bosonic and two fermionic super-projective invariants. In the two-loop vacuum case, the NS punctures are sewn in pairs while in the one-loop two-point case, two of the NS punctures $\mathcal{V}_1$ and $\mathcal{V}_2$ correspond to the position of vertex operators where external states are emitted (see Fig. 2.10).

We want to calculate the one loop diagram contributing to the two-point function for gluons, evaluated in the limit of vanishing momentum. The diagram must vanish to preserve gauge invariance: the reason is that if the diagram doesn't vanish, then the renormalized lagrangian will gain a mass term of the form $\frac{1}{2}m^2 A_\mu A^\mu$ which is not invariant under gauge transformations of the form $A_\mu \mapsto A_\mu + \partial_\mu \Lambda$ [82].

Let us consider the worldsheet superfield

$$\mathbf{X}^\mu(z|\theta) = X^\mu(z) + \mathrm{i}\theta\psi^\mu(z)\,. \tag{2.244}$$

The amplitude is calculated by inserting two vertex operators corresponding to the external gluon states. Since we want to show that the diagram vanishes, the overall normalization of the vertex operators is not important, and we can write them in terms of the superfield $\mathbf{X}^\mu$ as:

$$V[\mathbf{X}^\mu] \propto \epsilon_\mu D\mathbf{X}^\mu \mathrm{e}^{\mathrm{i}k\cdot\mathbf{X}} = \mathrm{i}\epsilon_\mu\left(\psi^\mu - \theta(\mathrm{i}\partial_z X^\mu + k\cdot\psi\psi^\mu)\right)\mathrm{e}^{\mathrm{i}k\cdot X} \tag{2.245}$$

where D is a the super-covariant derivative Eq. (2.84). The integral which we have to show vanishes is given in terms of the super-points $Z_1 = 1|\theta$ and $Z_2 = z|\phi$ by

$$\begin{aligned}\epsilon^1_\mu\epsilon^2_\nu \int \mathrm{d}z\,\mathrm{d}\theta\,\mathrm{d}\phi\;\langle D\mathbf{X}^\mu(Z_1)D\mathbf{X}^\nu(Z_2)\,\rangle \\ = \epsilon^1\cdot\epsilon^2\int \mathrm{d}z\,\mathrm{d}\theta\,\mathrm{d}\phi\, D_{Z_1}D_{Z_2}\mathcal{G}_N(Z_1,Z_2)\Big|_{\substack{Z_1=1|\theta\\ Z_2=z|\phi}}\end{aligned} \tag{2.246}$$

in terms of the worldsheet propagator with Neumann boundary conditions which is given by equation (25) of [9]

$$\mathcal{G}_N(Z,Y) = \log \mathcal{E}(X,Y) + \frac{1}{2}\sum_{\mu,\nu=1}^{g} \int_Y^Z \Omega_\mu (2\pi\, \mathrm{Im}\, \boldsymbol{\tau})^{-1}_{\mu\nu} \int_Z^Y \Omega_\nu \tag{2.247}$$

where $\mathcal{E}$ is the super-symmetric generalization of the prime form (Eq. (2.64)) expressed in the super-Schottky parametrization as:

$$\mathcal{E}(Z,Y) = (Z \doteq Y) \prod_\alpha{}' \frac{Z \doteq \mathbf{T}_\alpha(Y)}{Z \doteq \mathbf{T}_\alpha(Z)} \frac{Y \doteq \mathbf{T}_\alpha(Z)}{Y \doteq \mathbf{T}_\alpha(Y)}\,, \tag{2.248}$$

The notation $\prod'_\alpha$ means that the product is over all primary classes in the super Schottky group. A primary class is an equivalence class of primitive super Schottky group elements, *i.e.* those which cannot be written as a power of another super Schottky group element, $\mathbf{S}_\alpha \neq \mathbf{S}_\beta^n$ $(n \geq 2)$. Two primitive elements are in the same primary class if one is related to the other by cyclic permutation of its factors, or by inversion. Any h primary classes generate the h-loop super Schottky group.

$\boldsymbol{\Omega}_\mu$ is the supersymmetric generalization of the abelian differentials (see Eq. (2.238)) and $\boldsymbol{\tau}$ is the super-period matrix (see Eq. (2.241)). In our case, we have only one generator $\mathbf{S}_1(Z) = \mathbf{S}(z|\theta) = kz|k^{\frac{1}{2}}\theta$. Then we can calculate the sole abelian differential as

$$\Omega_1(Z) = \mathrm{d}Z D_Z \log \frac{Z \doteq U_\mu}{Z \doteq V_\mu} = \mathrm{d}Z D_Z \log Z + \mathcal{O}(k)\,, \tag{2.249}$$

since $\mathbf{T}_\alpha = \mathrm{Id}$ is the only super-Schottky group element whose right-most factor isn't $\mathbf{S}_1^{\pm 1}$. The sole entry in the super-period matrix is given by

$$2\pi \mathrm{Im} \boldsymbol{\tau}_{11} = -\log k + \mathcal{O}(k)\,. \tag{2.250}$$

Since there is only one generator for the Schottky group, the expression for the prime form becomes simply

$$\mathcal{E}(Z,Y) = (Z \doteq Y) \prod_{n>0} \frac{Z \doteq \mathbf{S}^n(Y)}{Z \doteq \mathbf{S}^n(Z)} \frac{Y \doteq \mathbf{S}^n(Z)}{Y \doteq \mathbf{S}^n(Y)} \tag{2.251}$$

while the other term that contributes to the propagator can be evaluated using Eq. (2.250) as

$$\frac{1}{2}\sum_{\mu,\nu=1}^{1} \int_Y^Z \Omega_\mu (2\pi \mathrm{Im} \boldsymbol{\tau})^{-1}_{\mu\nu} \int_Z^Y \Omega_\nu = \frac{1}{2\log k}\left(\log \frac{y}{z}\right)^2\,. \tag{2.252}$$

Putting these together into the expression Eq. (2.247) for the propagator, we find

$$\mathcal{G}_N(Z,Y) = \log \left[(Z \doteq Y) \prod_{n>0} \frac{Z \doteq \mathbf{S}^n(Y)}{Z \doteq \mathbf{S}^n(Z)} \frac{Y \doteq \mathbf{S}^n(Z)}{Y \doteq \mathbf{S}^n(Y)} \right] + \frac{1}{2\log k}\left(\log \frac{y}{z}\right)^2\,. \tag{2.253}$$

To calculate the double derivative of the propagator appearing in Eq. (2.246), we temporarily write $Z_i = z_i|\theta_i$ so that it makes sense to differentiate with respect to the bosonic part of Z_1 (which it doesn't if it's fixed equal to 1), then we find

$$D_{Z_1} D_{Z_2} \left(\log \frac{z_1}{z_2} \right)^2 = -2\theta_1\theta_2 \frac{1}{z_1 z_2} \tag{2.254}$$

and when reinstating the gauge-fixed values of Z_i and carrying out the integrals, this is just equal to

$$\int \mathrm{d}z\, \mathrm{d}\theta\, \mathrm{d}\phi\, D_{Z_1} D_{Z_2} \frac{1}{2\log k} \left(\log \frac{z_1}{z_2} \right)^2 \Big|_{\substack{Z_1 = 1|\theta \\ Z_2 = z|\phi}} = \frac{1}{\log k} \int_k^1 \mathrm{d} \log z = -1. \tag{2.255}$$

For the other term, we note

$$\begin{aligned} D_{Z_1} D_{Z_2} \log &\left[(Z_1 \dot{-} Z_2) \prod_{n>0} \frac{Z_1 \dot{-} \mathbf{S}^n(Z_2)}{Z_1 \dot{-} \mathbf{S}^n(Z_1)} \frac{Z_2 \dot{-} \mathbf{S}^n(Z_1)}{Z_2 \dot{-} \mathbf{S}^n(Z_2)} \right] \\ &= D_{Z_1} D_{Z_2} \log(Z_1 \dot{-} Z_2) + \sum_{n>0} \big[D_{Z_1} D_{Z_2} \log(Z_1 \dot{-} \mathbf{S}^n(Z_2)) \\ &\qquad + D_{Z_1} D_{Z_2} (\log Z_2 \dot{-} \mathbf{S}^n(Z_1)) \big] , \end{aligned} \tag{2.256}$$

where terms which are independent of one of either Z_1 or Z_2 vanish because they are annihilated by one of the two derivatives. Writing $\mathbf{S}^n(z|\theta) = k^n z | k^{\frac{n}{2}} \theta$, we have

$$D_{Z_1} D_{Z_2} \log(Z_1 \dot{-} \mathbf{S}^n(Z_2)) = \frac{1}{k^{-\frac{n}{2}} z_1 - k^{\frac{n}{2}} z_2 - \theta_1 \theta_2} , \tag{2.257}$$

then by swapping $Z_1 \leftrightarrow Z_2$ and using the fact that the D_{Z_i} are fermionic objects which anti-commute, we can use Eq. (2.257) to get a similar expression for $D_{Z_1} D_{Z_2} \log(Z_2 \dot{-} \mathbf{S}^n(Z_1))$, then both can be submitted into Eq. (2.256) to get an expression for the double derivative of the super prime form,

$$D_{Z_1} D_{Z_2} \log \mathcal{E}(Z_1, Z_2) = \sum_{n \in \mathbf{Z}} \frac{1}{k^{-\frac{n}{2}} z_1 - k^{\frac{n}{2}} z_2 - \theta_1 \theta_2} , \tag{2.258}$$

so the integral to be evaluated is now

$$\int \mathrm{d}z\, \mathrm{d}\theta\, \mathrm{d}\phi\, \left(D_{Z_1} D_{Z_2} \log \mathcal{E}(Z_1, Z_2) \right) \Big|_{\substack{Z_1 = 1|\theta \\ Z_2 = z|\phi}} = \sum_{n \in \mathbf{Z}} \int \mathrm{d}z\, \mathrm{d}\theta\, \mathrm{d}\phi\, \frac{1}{k^{-\frac{n}{2}} - k^{\frac{n}{2}} z - \theta\phi} . \tag{2.259}$$

Now let's say we naïvely ignore the need to fix the right bosonic variables and choose the integration region to be $z \in [k, 1]$. Then we get

$$\sum_{n \in \mathbf{Z}} \int_{z=k}^{z=1} \mathrm{d}z\, \mathrm{d}\theta\, \mathrm{d}\phi\, \frac{1}{k^{-\frac{n}{2}} - k^{\frac{n}{2}} z - \theta\phi} = \sum_{n \in \mathbf{Z}} \left[\frac{1}{1 - k^n z} \right]_k^1 \tag{2.260}$$

This expression is ill-defined because the $n = 0$ and $n = -1$ terms contains a division by 0 from the $z = 1$ and $z = k$ boundaries, respectively. Even if we use a cut-off to regularize by changing the integration region for the bosonic modulus to $z \in [k+\epsilon, 1-\epsilon]$, then the partial sums are given by

$$S_N = \sum_{n=-N}^{N} \left(\frac{1}{1-k^n(1-\epsilon)} - \frac{1}{1-k^n(k+\epsilon)} \right) = \frac{k^N+1}{k^N-k} + \frac{1+k}{\epsilon} + \mathcal{O}(\epsilon^0)\,. \qquad (2.261)$$

which still diverges in the limit $\epsilon \to 0$.

The proper approach to computing the integral is to consider the integrand $\boldsymbol{\nu}$ as an *integral form* which is the only type of mathematical object that can be meaningfully integrated on a supermanifold (see [83] for an explanation of this fact) such as the supermoduli space $\widehat{\mathfrak{M}}^{\mathrm{os}}_{1,2}$ and rewrite it as an exterior derivative, $\boldsymbol{\nu} = \mathrm{d}\boldsymbol{\omega}$. The integral should then be evaluated via the supermanifold version of Stokes' theorem by evaluating the integral of $\boldsymbol{\omega}$ over the *boundary* of supermoduli space, $\partial\widehat{\mathfrak{M}}^{\mathrm{os}}_{1,2}$. Formally, the result of rewriting the right-hand-side of Eq. (2.259) with Stokes' theorem is

$$\sum_{n\in\mathbf{Z}} \int \mathrm{d}z\,\mathrm{d}\theta\,\mathrm{d}\phi\, \frac{1}{k^{-\frac{n}{2}} - k^{\frac{n}{2}} z - \theta\phi} = \sum_{n\in\mathbf{Z}} (-k^{-\frac{n}{2}}) \int_{\partial} \mathrm{d}\theta\,\mathrm{d}\phi\, \log\left(k^{-\frac{n}{2}} - k^{\frac{n}{2}} z - \theta\phi\right)\,. \qquad (2.262)$$

The purely fermionic integral at the $0|2$-dimensional boundary is dependent upon the the choice of which bosonic variable is held fixed, and therefore we should express the integrand in terms of the appropriate variables for each boundary. Good bosonic variables to fix are those which vanish (or diverge to infinity) at the boundary. We can define our upper limit of integration as the locus $Z_2 = Z_1$ and our lower limit of integration as the locus $Z_2 = \mathbf{S}(Z_1)$, so we can define two new bosonic moduli:

$$Y_0 = Z_2 \dot{-} Z_1 \qquad\qquad Y_1 = Z_2 \dot{-} \mathbf{S}(Z_1) \qquad (2.263)$$
$$= (z-1+\theta\phi, \phi-\theta)\,; \qquad\qquad = (z-k+k^{\frac{1}{2}}\theta\phi, \phi-k^{\frac{1}{2}}\theta)\,, \qquad (2.264)$$

whose bosonic parts

$$y_0 = z-1+\theta\phi\,; \qquad\qquad y_1 = z-k+k^{\frac{1}{2}}\theta\phi \qquad (2.265)$$

define the two boundary components via $y_1 = 0$ and $y_0 = 0$. Expressing z in terms of these variable, we can re-write the integrand as either

$$\log\left(k^{-\frac{n}{2}} - k^{\frac{n}{2}} z - \theta\phi\right) = \log(k^{-\frac{n}{2}} - (1+y_0)k^{\frac{n}{2}}) + \frac{\theta\phi(k^{\frac{n}{2}}-1)}{k^{-\frac{n}{2}} - k^{\frac{n}{2}}(y_0+1)} \qquad (2.266)$$

or

$$= \log(k^{-\frac{n}{2}} - (k+y_1)k^{\frac{n}{2}}) + \frac{\theta\phi(k^{\frac{n+1}{2}}-1)}{k^{-\frac{n}{2}} - k^{\frac{n}{2}}(y_1+k)}\,, \qquad (2.267)$$

near $y_0 = 0$ and $y_1 = 0$, respectively. Let us evaluate the fermionic integral at the two boundary components initially with a small cutoff $y_0 = y_1 = \epsilon$ to avoid an infinite $\log(0)$ divergence, although it will simply be annihilated by the Berezin integral. Writing down the difference of the two boundary components, we get

$$\int_\partial \mathrm{d}\theta\,\mathrm{d}\phi\,\log\left(x^{-n}-x^n z-\theta\phi\right) = -\lim_{\epsilon\to 0}\left(\frac{k^{\frac{n}{2}}-1}{k^{-\frac{n}{2}}-k^{\frac{n}{2}}(\epsilon+1)}-\frac{k^{\frac{n+1}{2}}-1}{k^{-\frac{n}{2}}-k^{\frac{n}{2}}(\epsilon+k)}\right) \tag{2.268}$$

$$= \begin{cases} -\frac{1}{1+k^{\frac{1}{2}}} & n=0 \\ \frac{1}{1+k^{\frac{1}{2}}} & n=-1 \\ \frac{k^{\frac{n}{2}}}{1+k^{\frac{n}{2}}}-\frac{k^{\frac{n}{2}}}{1+k^{\frac{1+n}{2}}} & \text{otherwise}\,. \end{cases} \tag{2.269}$$

Summing this up multiplied term-by-term with the factor of $(-k^{-\frac{n}{2}})$ in Eq. (2.262), we arrive at

$$\int \mathrm{d}z\,\mathrm{d}\theta\,\mathrm{d}\phi\,\left(D_{Z_1}D_{Z_2}\log\mathcal{E}(Z_1,Z_2)\right)\Big|_{\substack{Z_1=1|\theta\\ Z_2=z|\phi}} = -\sum_{n\in\mathbf{Z}}\left(\frac{1}{1+k^{\frac{n}{2}}}-\frac{1}{1+k^{\frac{1+n}{2}}}\right)\,. \tag{2.270}$$

The partial sums are given by

$$-\sum_{n=-N}^{N}\left(\frac{1}{1+k^{\frac{n}{2}}}-\frac{1}{1+k^{\frac{1+n}{2}}}\right) = -\frac{k^{\frac{N}{2}}}{1+k^{\frac{N}{2}}}+\frac{1}{1+k^{\frac{N+1}{2}}}\,. \tag{2.271}$$

Since $|k^{\frac{1}{2}}|<1$ this converges as $N\to\infty$, so Eq. (2.270) is simply equal to 1. Adding this to 2.255, which we calculated as -1, we see that the super-moduli space integral of the double derivative of the propagator is given by

$$\int \mathrm{d}z\,\mathrm{d}\theta\,\mathrm{d}\phi\,\left(D_{Z_1}D_{Z_2}\log\mathcal{G}_N(Z_1,Z_2)\right)\Big|_{\substack{Z_1=1|\theta\\ Z_2=z|\phi}} = 0\,, \tag{2.272}$$

so the diagram vanishes and gauge invariance is preserved.

Note that this argument does not hold if the sector of the worldsheet theory under consideration has Dirichlet boundary conditions instead of Neumann boundary conditions in that case the worldsheet propagator is no longer given by $\mathcal{G}_N$ as in Eq. (2.247), but rather by the Dirichlet propagator which only depends on the prime form:

$$\mathcal{G}_D(Z,Y) = \log\mathcal{E}(X,Y)\,. \tag{2.273}$$

In this case, the -1 from Eq. (2.255) no longer appears to cancel the 1 from Eq. (2.270), which means that the one-loop correction to the two-point function is finite and non-zero. However, this is not a problem as it would have been with Neumann boundary conditions, since Dirichlet boundary conditions correspond to scalar fields in the spacetime effective theory which are not described by a gauge invariant Lagrangian and therefore may undergo mass renormalization without a problem; in this case the mass renormalization will be of the order g^2/α'.

Chapter 3

Strings in background fields

3.1 Free open strings in a constant background magnetic field

In this section we calculate the effect of the background magnetic field on the worldsheet theory; the original calculations were done for bosonic strings in [84, 85, 86] and [87]. Superstring effective theories for spacetime gauge fields were studied in [88, 89] and [90]. We follow the calculation in [91, 92] and chapters 16 and 19 of [19]. Consider an open bosonic string propagating in a fixed background Kalb-Ramond field. The string can be described by the Euclidean worldsheet action in conformal gauge as

$$S_{\mathrm{bos}} = -\frac{1}{4\pi\alpha'}\int_{-\infty}^{\infty}\mathrm{d}\tau\int_0^{\pi}\mathrm{d}\sigma\left(\partial^\alpha X^M\partial_\alpha X^N G_{MN} + \mathrm{i}\,\epsilon^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N B_{MN}\right). \qquad (3.1)$$

where $\epsilon^{\alpha\beta}$ is the antisymmetric symbol with $\epsilon^{\sigma\tau}=1$. This action is not, however, gauge invariant: under a gauge transformation $B_{MN}\to B_{MN}+\partial_M\Lambda_N-\partial_N\Lambda_M$, it transforms as $S_{\mathrm{bos}}\to S_{\mathrm{bos}}+\delta S_{\mathrm{bos}}$, where

$$\delta S_{\mathrm{bos}} = -\frac{1}{2\pi\alpha'}\int_{-\infty}^{\infty}\mathrm{d}\tau\int_0^{\pi}\mathrm{d}\sigma\,\mathrm{i}\,\partial_\sigma X^M\partial_\tau X^N\left(\partial_M\Lambda_N-\partial_N\lambda_M\right). \qquad (3.2)$$

We can rewrite δS in terms of total derivatives with the use of the chain rule: $\partial_\alpha X^R\partial_R\Lambda^S = \partial_\alpha\Lambda^S$. The contribution from the τ derivative vanishes when we assume the gauge transformation tends to 0 as $\tau\to\pm\infty$, while the contribution from the σ derivative gives

$$\delta S_{\mathrm{bos}} = -\frac{1}{2\pi\alpha'}\int_{-\infty}^{\infty}\mathrm{d}\tau\left[\mathrm{i}\,\Lambda_M\partial_\tau X^M\right]_{\sigma=0}^{\pi}, \qquad (3.3)$$

which doesn't vanish in general. Gauge invariance can be imposed by coupling the endpoints of the string to the U(1) gauge field A_μ of the D-brane to which they are attached by adding a boundary term to the action: $S_{\mathrm{bos}}\to S'_{\mathrm{bos}}=S_{\mathrm{bos}}+S_{\partial\,\mathrm{bos}}$, where

$$S_{\partial\,\mathrm{bos}} = \mathrm{i}\int_{\sigma=0}\mathrm{d}X^M A_M - \mathrm{i}\int_{\sigma=\pi}\mathrm{d}X^M A_M\,. \qquad (3.4)$$

The action is then invariant under the combined gauge transformation

$$B_{MN} \to B_{MN} + \partial_M \Lambda_N - \partial_N \Lambda_M \,; \qquad\qquad A_M \to A_M - \frac{\Lambda_M}{2\pi\alpha'} \,. \tag{3.5}$$

Note that since this gauge transformation is not necessarily of the form $\Lambda = \mathrm{d}\phi$, the D-brane world-volume U(1) field strength $F = \mathrm{d}A$ is not gauge invariant in general; only the combination $\mathcal{F} = B + 2\pi\alpha' F$ is.

We can find the equation of motion for the action S'_{bos} by solving $\delta S'_{\text{bos}} = 0$. Under a variation $X^M \to X^M + \delta X^M$ which vanishes at $\tau \to \pm\infty$, the bulk part of the action varies as $S_{\text{bos}} \to S_{\text{bos}} + \delta S_{\text{bos}}$ where

$$\begin{aligned} \delta S_{\text{bos}} = &-\frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} \mathrm{d}\tau \, \Big[\delta X^M \big(G_{MN} \partial_\sigma X^N + \mathrm{i}\, B_{MN} \partial_\tau X^N \big) \Big]_{\sigma=0}^{\pi} \\ &+ \frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} \mathrm{d}\tau \int_0^{\pi} \mathrm{d}\sigma \, \delta X^M \, \partial_\alpha \partial^\alpha X_M \,; \end{aligned} \tag{3.6}$$

similarly the coupling between the endpoints and the U(1) gauge field A_μ varies as $S_{\partial\,\text{bos}} \to S_{\partial\,\text{bos}} + \delta S_{\partial\,\text{bos}}$ with

$$\delta S_{\partial\,\text{bos}} = -\mathrm{i}\, \delta \int_{-\infty}^{\infty} \mathrm{d}\tau \, \Big[A_M \partial_\tau X^\mu \Big]_{\sigma=0}^{\pi} = -\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}\tau \, \Big[\delta X^M F_{MN} \partial_\tau X^N \Big]_{\sigma=0}^{\pi} \,, \tag{3.7}$$

where we've used $\delta A_M = \partial_N A_M \delta X^N$. By the fundamental lemma of the calculus of variations, the equation of motion can be read off from the second line of Eq. (3.6) as $\partial_\alpha \partial^\alpha X_M = 0$. For the "Dirichlet directions" transverse to the D-branes, we have $0 = \delta X^I|_{\sigma=0,\pi}$ so the boundary contribution vanishes automatically. In the "Neumann directions" this doesn't hold but the boundary contribution coming from Eq. (3.7) and the first line of Eq. (3.6) must still vanish, so we get the boundary condition

$$\big(G_{\mu\nu} \partial_\sigma + \mathrm{i}\, \mathcal{F}_{\mu\nu} \partial_\tau \big) X^\nu \Big|_{\sigma=0,\pi} = 0 \,. \tag{3.8}$$

We can rewrite everything in terms of the complex worldsheet coordinates Eq. (2.28); then Eq. (3.8) becomes

$$\big((G_{\mu\nu} + \mathcal{F}^{(\sigma)}_{\mu\nu})\, \partial - (G_{\mu\nu} - \mathcal{F}^{(\sigma)}_{\mu\nu})\, \overline{\partial} \big) X^\nu \Big|_{\mathrm{Im}(z)=0} = 0 \,. \tag{3.9}$$

where we've used $z = \overline{z}$. We have $\partial_\sigma \partial_\sigma + \partial_\tau \partial_\tau = 4|z|^2\, \overline{\partial}\partial$ so the equation of motion becomes $\overline{\partial}\partial X^\mu = 0$. The boundary conditions Eq. (3.9) can be recast in terms of a 'reflection matrix' $\mathcal{R}^{(\sigma)}$ as

$$\overline{\partial} X^\mu \big|_{\mathrm{Im}(z)=0} = \big(R^{(\sigma)} \big)^\mu{}_\nu \, \partial X^\nu \big|_{\mathrm{Im}(z)=0} \,; \qquad R^{(\sigma)} = \big(G - \mathcal{F}^{(\sigma)} \big)^{-1} \big(G + \mathcal{F}^{(\sigma)} \big) \,. \tag{3.10}$$

We can solve the equation of motion with these boundary conditions in terms of chiral fields $y^\mu(z, \overline{z}) = y^\mu(z)$ which are sections of a holomorphic vector bundle on $\mathbb{C} \setminus \{0\}$ with

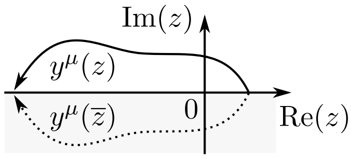


Figure 3.1: Complex conjugate paths between the positive and negative real axes.

non-trivial monodromy around $z = 0$ given by a 'monodromy matrix' R:

$$y^\mu(\mathrm{e}^{2\pi\mathrm{i}} z) = R^\mu{}_\nu \, y^\nu(z)\,; \qquad\qquad R = \big(R^{(\pi)}\big)^{-1}\big(R^{(0)}\big)\,, \tag{3.11}$$

by writing

$$X^\mu(z,\overline{z}) = q^\mu + \frac{1}{2}\Big(y^\mu(z) + \big(R^{(0)}\big)^\mu{}_\nu \, y^\nu(\overline{z})\Big)\,. \tag{3.12}$$

This clearly satisfies Eq. (3.10) at the boundary $\sigma = 0$. To verify that the boundary condition as $\sigma = \pi$ is satisfied, we consider $\overline{\partial} y^\mu(\overline{z})$ and $\partial y^\mu(z)$ evaluated at a point on the negative real axis, where the branch is defined by requiring continuity for z in the upper-half-plane. As z moves along a path in the upper-half plane, $\overline{z}$ will move along the complex conjugate path in the lower-half plane, and therefore when the paths meet at the negative real axis, $\partial y^\mu(z)$ will be on a different branch from $\overline{\partial} y^\mu(\overline{z})$ (see Fig. 3.1), so we have

$$\partial y^\mu(z)\big|_{\mathrm{Im}(z)=0} = R^\mu{}_\nu \, \overline{\partial} y^\nu(\overline{z})\big|_{\mathrm{Im}(z)=0}\,, \tag{3.13}$$

By inserting the expression for R in Eq. (3.11) we see that this satisfies Eq. (3.10) for $\sigma = \pi$; therefore X^μ in Eq. (3.12) is a classical solution in this background.

Now let's assume a flat metric, $G_{\mu\nu} = \eta_{\mu\nu}$ and assume that the background fields $\mathcal{F}$ are magnetic fields in the same plane as each other, *i.e.* the x^1–x^2 plane. Let's work in a Lorentz frame in which $\mathcal{F}^{(\sigma)}_{\mu\nu} = f^{(\sigma)}(\eta_{\mu 1}\eta_{2\nu} - \eta_{\mu 2}\eta_{\nu 1})$, then the reflection matrices Eq. (3.10) take the form

$$R^{(\sigma)} = \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & \frac{1-f^2_{(\sigma)}}{1+f^2_{(\sigma)}} & \frac{-2f_{(\sigma)}}{1+f^2_{(\sigma)}} & \\ 0 & \frac{2f_{(\sigma)}}{1+f^2_{(\sigma)}} & \frac{1-f^2_{(\sigma)}}{1+f^2_{(\sigma)}} & \\ \hline & & & \mathbf{1} \end{array}\right)\,. \tag{3.14}$$

This is just a rotation in the X^1–X^2 plane since we can write

$$\frac{1-f^2_{(\sigma)}}{1+f^2_{(\sigma)}} = \cos(2\pi\theta^{(\sigma)})\,; \quad \frac{-2f_{(\sigma)}}{1+f^2_{(\sigma)}} = \sin(2\pi\theta^{(\sigma)})\,, \quad \text{where} \quad \pi\theta^{(\sigma)} \equiv -\arctan f_{(\sigma)}\,. \tag{3.15}$$

$R^{(\sigma)}$ can be diagonalized by grouping pairs of target spacetime coordinates X^μ together into a complex basis $(Z^i, \overline{Z}^i)$, with

$$Z^n = X^{2i-1} - \mathrm{i}\, X^{2i}\,, \qquad n = 1, \ldots, D/2 - 1\,; \qquad Z^{D/2} = X^0 - \mathrm{i}\, X^D\,, \tag{3.16}$$

and $\overline{Z}^i \equiv \left(Z^i\right)^*$. In this basis, the reflection matrices are given by

$$R^{(\sigma)} Z^1 = \mathrm{e}^{-2\pi \mathrm{i}\theta^{(\sigma)}} Z^1\,; \qquad R^{(\sigma)} \overline{Z}^1 = \mathrm{e}^{2\pi \mathrm{i}\theta^{(\sigma)}} \overline{Z}^1\,; \tag{3.17}$$

$$R^{(\sigma)} Z^i = Z^i\,; \qquad R^{(\sigma)} \overline{Z}^i = \overline{Z}^i\,, \qquad i \neq 1\,. \tag{3.18}$$

and therefore the monodromy matrix Eq. (3.11) satisfies:

$$R Z^1 = \mathrm{e}^{2\pi \mathrm{i}\left(\theta^{(\pi)} - \theta^{(0)}\right)} Z^1\,; \qquad R \overline{Z}^1 = \mathrm{e}^{-2\pi \mathrm{i}\left(\theta^{(\pi)} - \theta^{(0)}\right)} \overline{Z}^1\,, \tag{3.19}$$

and acts trivially on the other directions.

Now, we have seen that for the free bosonic string parametrized by the upper-half-plane, the analytic function ∂X^μ can be expanded as a Laurent series in powers of z according to Eq. (2.132). In the case of strings with a monodromy $\theta \equiv \theta^{(\pi)} - \theta^{(0)}$, the holomorphic function ∂X^μ can be expanded as a series multiplying $z^{-n+\theta}$, which has the requisite monodromy properties for all n. To be precise, we use the expansion (Eq. (2.12) of [92])

$$\partial Z^i = -\mathrm{i}\sqrt{2\alpha'}\Big(\sum_{n=1}^{\infty} \bar{a}^i_{n-\theta_i} z^{-n+\theta_i - 1} + \sum_{n=0}^{\infty} a^{\dagger i}_{n+\theta_i} z^{n+\theta_i - 1} \Big)\,, \tag{3.20}$$

$$\partial Z^i = -\mathrm{i}\sqrt{2\alpha'}\Big(\sum_{n=0}^{\infty} a^i_{n+\theta_i} z^{-n+\theta_i - 1} + \sum_{n=1}^{\infty} \bar{a}^{\dagger i}_{n-\theta_i} z^{n+\theta_i - 1} \Big)\,. \tag{3.21}$$

The commutation relations Eq. (2.134) need to be modified as follows: we have

$$[\bar{a}^i_{n-\theta_i}, \bar{a}^{\dagger j}_{m-\theta_j}] = (n - \theta_i)\delta^{ij}\delta^{ij}\delta_{n,m} \qquad \text{for } n, m \geq 1\,, \tag{3.22}$$

$$[a^i_{n+\theta_i}, a^{\dagger j}_{m+\theta_j}] = (n + \theta_i)\delta^{ij}\delta_{n,m} \qquad \text{for } n, m \geq 0\,; \tag{3.23}$$

there is a 'twisted vacuum' [92] $|\Theta\rangle$ for which some of these are annihilation operators satisfying $\bar{a}^i_{n-\theta_i}|\Theta\rangle = a^i_{m+\theta_i}|\Theta\rangle = 0$ for $n \geq 1$ and $m \geq 0$ while the rest are creation operators.

3.1.1 Superstrings

We can extend the construction to strings with worldsheet fermions by appending the action S_{bos} in Eq. (3.1) with the action[92]

$$S_{\text{ferm}} = -\frac{\mathrm{i}}{4\pi\alpha'} \int_{-\infty}^{\infty} \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma\, \overline{\chi}^M \rho^\alpha \partial_\alpha \chi^N (G_{MN} + B_{MN})\,, \tag{3.24}$$

where $\{\rho^\alpha\}$ are a basis for a two-dimensional representation of the worldsheet Clifford algebra, and $\overline{\chi}^M = (\chi^M)^\dagger \rho^\tau$. As before, this is not gauge invariant and the combination

$S_{\text{bos}} + S_{\partial\,\text{bos}} + S_{\text{ferm}}$ is not supersymmetric, therefore we need to include a boundary term [93], $S_{\text{ferm}} \to S'_{\text{ferm}} = S_{\text{ferm}} + S_{\partial\,\text{ferm}}$, where

$$S_{\partial\,\text{ferm}} = -\frac{\mathrm{i}}{2}\int_{-\infty}^{\infty} \mathrm{d}\tau \left[\overline{\chi}\,\rho^\tau \chi^N F^{(\sigma)}_{MN}\right]^{\pi}_{\sigma=0}. \tag{3.25}$$

In terms of the chiral spinors $\chi_\pm$ defined in Eq. (2.67) and the monodromy matrices $R^{(\sigma)}$ defined in Eq. (3.10), the boundary conditions become[92]

$$\chi^M_-\big|_{\sigma=0} = \left(R^{(0)}\right)^M{}_N \chi^N_+\big|_{\sigma=0} \qquad \text{and } \chi^M_-\big|_{\sigma=\pi} = -\eta\left(R^{(\pi)}\right)^M{}_N \chi^N_+\big|_{\sigma=\pi}\,, \tag{3.26}$$

where $\eta = 1$ for the NS sector and $\eta = -1$ for the R sector. In the spacetime basis in which the monodromy matrix is diagonal, in terms of the upper-half-plane parametrization of the open string worldsheet, we can solve Eq. (3.26) in terms of sections of a spin bundle on the worldsheet with nontrivial monodromy satisfying

$$\Psi^i(\mathrm{e}^{2\pi\mathrm{i}}z) = \eta\mathrm{e}^{2\pi\mathrm{i}\theta}\Psi^i(z)\,; \qquad \bar\Psi^i(\mathrm{e}^{2\pi\mathrm{i}}z) = \eta\mathrm{e}^{-2\pi\mathrm{i}\theta}\bar\Psi^i(z)\,. \tag{3.27}$$

with the mode expansion

$$\Psi^i(z) = \sqrt{2\alpha'}\sum_{n=\nu}^{\infty}\left(\bar\Psi^i_{n-\theta_i} z^{-n-\theta_i-\frac12} + \Psi^{\dagger i}_{n+\theta_i} z^{n+\theta_i-\frac12}\right), \tag{3.28}$$

$$\bar\Psi^i(z) = \sqrt{2\alpha'}\sum_{n=\nu}^{\infty}\left(\Psi^i_{n+\theta_i} z^{-n-\theta_i-\frac12} + \bar\Psi^{\dagger i}_{n-\theta_i} z^{n-\theta_i-\frac12}\right), \tag{3.29}$$

where $\nu = 0$ for the R sector and $\nu = \frac12$ for the NS sector. The modes satisfy the anti-commutation relations [92]

$$\{\Psi^i_{n+\theta_i}, \Psi^{\dagger i}_{m+\theta_i}\} = \{\bar\Psi^i_{n-\theta_i}, \bar\Psi^{\dagger i}_{m-\theta_i}\} = \delta^{ij}\delta_{nm} \text{ for } n, m \geq \nu\,. \tag{3.30}$$

Ψ and $\bar\Psi$ are related to $\chi_\pm$ via $\chi^M_+(z) = z^{\frac12}\psi^M_+(z)$ and $\chi^M_-(\bar z) = \bar z^{\frac12}\psi^M_-(\bar z)$.

3.2 Higher loop string diagrams in background fields

Free strings propagating in a constant background field can be described by giving the $\mathbf{X}^\mu(z|\theta) = X^\mu(z) + \mathrm{i}\theta\psi^\mu(z)$ superfield a non-trivial monodromy around $z = 0$ on the complex plane.

For higher-loop amplitudes, the situation is similar. A h-loop open string worldsheet has $(h+1)$ boundaries, potentially attached to $(h+1)$ different D branes, but the $\mathbf{X}^\mu$ superfields living on the worldsheet are sensitive only to the difference between the strengths of the background fields living on each D brane—assuming that the spacetime reflection matrices associated to each boundary commute with each other, which is an assumption we are making.

For the two-loop vacuum amplitudes which we are computing, the string worldsheets can end on (up to) three D branes, each of which supports a background field of the form

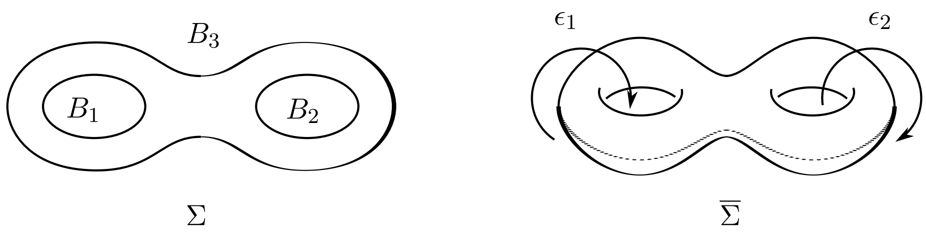


Figure 3.2: Open string worldsheets coupled to D-branes with background fields have compact Riemann surfaces with non-trivial monodromies as their doubles.

Eq. (4.1). Grouping the worldsheet fields $\mathbf{X}^1$ and $\mathbf{X}^2$ together as $\mathbf{Z}^{\pm} = \frac{1}{\sqrt{2}}(\mathbf{X}^1 \pm \mathrm{i}\mathbf{X}^2)$, we find that the worldsheet field $\mathbf{Z}$ on the *double* of the bordered Riemann surface (which is a genus $h = 2$ compact Riemann surface) is a section of a line bundle with non-trivial monodromies around the two b_i-cycles. To be precise, if the three open string worldsheet boundaries have background fields of the form in Eq. (4.1) whose strength is given by B_i as indicated in the first diagram in Fig. 3.2, then the worldsheet fields $\mathbf{Z}^{\pm}$ will have monodromy $\mathrm{e}^{\pm 2\pi\mathrm{i}\epsilon_i}$ around the homology cycles on the double surface as indicated in the second diagram of Fig. 3.2, where the relation between B_i and ϵ_j is given by

$$\tan(\pi\epsilon_1) = 2\pi\alpha'(B_1 - B_3)\,; \qquad\qquad \tan(\pi\epsilon_2) = 2\pi\alpha'(B_3 - B_2)\,. \tag{3.31}$$

In particular, for the $\mathbf{Z}^{\pm}$ sector of the worldsheet theory the abelian differentials need to be modified to account for the twists; they should be replaced with *Prym differentials* which are holomorphic sections of a line bundle on the SRS with the same non-trivial monodromies around the homology cycles. Since the period matrix is defined in terms of the abelian differentials, it too will have to be modified.

The twisted period matrix on a Riemann surface

It was shown how to write down Prym differentials for Riemann surfaces in terms of the Schottky group using the sewing procedure in [51] and [16]; the periods of these differentials were first studied in [16]; we compute them with a different method used in [94] which we will modify for use with superstrings.

The space of Prym differentials is $(h-1)$-dimensional on a surface with h handles, so the twisted period matrix will be an $(h-1)\times(h-1)$ matrix. In our case, $h = 2$ so the twisted period matrix is just one number, which can be shown to be given by the expression [94]

$$\tau_{\vec{\epsilon}} = \left\{ \frac{\mathrm{e}^{-\mathrm{i}\pi(\epsilon_1+\epsilon_2)}}{\sin(\pi(\epsilon_1+\epsilon_2))} \int_0^{\eta} \mathrm{e}^{2\pi\mathrm{i}\Delta^z\cdot\vec{\epsilon}}(1 - \mathrm{e}^{2\pi\mathrm{i}(\vec{\epsilon}\cdot\tau)_2})\,\Omega_1^{\vec{\epsilon}\cdot\tau}(z) \right\} + (\epsilon_\mu \to -\epsilon_\mu)\,. \tag{3.32}$$

where the monodromies around the a-cycles are trivial and the monodromies around the b-cycles are given by ϵ_μ, and the two Schottky generators are taken to have fixed points

$(\eta_1, \xi_1) = (0, \infty)$ and $(\eta_2, \xi_2) = (\eta, 1)$. In Eq. (3.32), Δ^z_μ is the vector of Riemann constants and $\Omega_1^{\vec{\epsilon}\cdot\tau}$ is the Prym differential with monodromy $(\vec{\epsilon}\cdot\tau)_\mu$ along the a_μ homology cycle. Here τ is the period matrix.

Δ^z_μ can be expressed in the Schottky parametrization as (Eq. (A.21) of [8])

$$\Delta^z_\mu = \frac{1}{2\pi\mathrm{i}}\left\{-\frac{1}{2}\log k_\mu - \pi\mathrm{i} + \sum_{\nu=1}^{g}{}^{(\nu)}\sum_{\alpha}{}^{(\mu)}\log\frac{\xi_\nu - T_\alpha(\eta_\mu)}{\xi_\nu - T_\alpha(\xi_\mu)}\frac{z - T_\alpha(\xi_\mu)}{z - T_\alpha(\eta_\mu)}\right\} \tag{3.33}$$

where the second sum ${}^{(\nu)}\sum_\alpha{}^{(\mu)}$ is over all elements of the Schottky group which have neither $S_\nu^{\pm 1}$ as their left-most element nor $S_\mu^{\pm 1}$ as their right-most element.

A basis of $(h-1)$ Prym differentials with monodromy ϵ_μ along the a_μ homology cycle can be expressed as (Eq. (3.11) of [94])

$$\Omega^{\vec{\epsilon}}_\nu(z) = \zeta^{\vec{\epsilon}}_\nu(z) - \frac{1-\mathrm{e}^{2\pi\mathrm{i}\epsilon_\nu}}{1-\mathrm{e}^{2\pi\mathrm{i}\epsilon_h}}\zeta^{\vec{\epsilon}}_h(z) \qquad\qquad \nu = 1,\ldots,(h-1)\,. \tag{3.34}$$

Here $\zeta^{\vec{\epsilon}}_\mu$ are a set of h 1-forms with the same monodromy, which are holomorphic everywhere except some arbitrary base point z_0, defined as (Eq. (3.15) of [94])

$$\begin{aligned}\zeta^{\vec{\epsilon}}_\mu(z) = \Big(\sum_{\alpha}{}^{(\mu)}\mathrm{e}^{2\pi i(\vec{\epsilon}\cdot N_\alpha + \epsilon_\mu)}\Big[\frac{1}{z-T_\alpha(\eta_\mu)} - \frac{1}{z-T_\alpha(\xi_\mu)}\Big] \\ + (1-\mathrm{e}^{2\pi i\epsilon_\mu})\sum_\alpha \mathrm{e}^{2\pi i\vec{\epsilon}\cdot N_\alpha}\Big[\frac{1}{z-T_\alpha(z_0)} - \frac{1}{z-T_\alpha(\alpha^\alpha_\mu)}\Big]\Big)\mathrm{d}z\end{aligned} \tag{3.35}$$

where the first sum is over all Schottky group elements which don't have $S_\mu^{\pm\ell}$ as their right-most factor and the second sum is over all Schottky group elements. Also in Eq. (3.35),

$$a^\alpha_\mu = \begin{cases}\eta_\mu & \text{if } T_\alpha = T_\beta S^\ell_\mu \text{ with } \ell \geq 1\\ \xi_\mu & \text{otherwise,}\end{cases} \tag{3.36}$$

and N^μ_α counts how many times each Schottky generator appears in T_α, defined by $N^\mu_\alpha = 0$ for $T_\alpha = \mathrm{Id}$ and $N^\mu_\alpha = N^\mu_\beta \pm 1$ for $T_\alpha = S_\mu^{\pm 1}T_\beta$.

Note that Ω^ϵ_μ defined in Eq. (3.34) is not the same as $\Omega^{\epsilon\cdot\tau}_\mu$ which appears in Eq. (3.32); to get that we need to replace ϵ_μ with $(\epsilon\cdot\tau)_\mu$ everywhere it appears.

3.2.1 The twisted period matrix on a super Riemann surface

To supersymmetrize $\tau_{\vec{\epsilon}}$, we find supersymmetric extensions of Δ^z_μ and Ω^ϵ_μ, replace the period matrix τ with the super-period matrix $\boldsymbol{\tau}$ in the phases $\mathrm{e}^{2\pi\mathrm{i}(\epsilon\cdot\tau)_\mu}$, and replace the integration between the fixed points $0 = \eta_1$ and $\eta = \eta_2$ with an integration between the super fixed points $0|0 = \mathbf{u}_1$ and $u|\theta = \mathbf{u}_2$.

To supersymmetrize Δ^z_μ, we need to replace the cross-ratios in Eq. (3.33) with super-projective invariant cross-ratios, and replace the fixed points η_μ, ξ_μ with $\mathbf{u}_\mu = u_\mu|\theta_\mu$, $\mathbf{v}_\mu = v_\mu|\phi_\mu$, and replace the base point z_0 with a super-point $\mathbf{z} = z|\psi$. The formula becomes, then,

$$\mathbf{\Delta}_{\mu}^{\mathbf{z}} = \frac{1}{2\pi\mathrm{i}}\left\{-\frac{1}{2}\log k_{\mu} - \pi\mathrm{i} + \sum_{\nu=1}^{g}{}^{(\nu)}\sum_{\alpha}{}^{(\mu)}\log\frac{\langle\mathbf{v}_{\nu}|\mathbf{T}_{\alpha}|\mathbf{u}_{\mu}\rangle}{\langle\mathbf{v}_{\nu}|\mathbf{T}_{\alpha}|\mathbf{v}_{\mu}\rangle}\frac{\langle\mathbf{z}|\mathbf{T}_{\alpha}|\mathbf{v}_{\mu}\rangle}{\langle\mathbf{z}|\mathbf{T}_{\alpha}|\mathbf{u}_{\mu}\rangle}\right\}. \tag{3.37}$$

For our purposes, we want to compute $\mathbf{\Delta}_{\mu}^{\mathbf{z}}$ for $h = 2$ with the following fixed points:

$$\mathbf{u}_1 = 0|0\,; \qquad \mathbf{v}_1 = \infty|0\,; \qquad \mathbf{u}_2 = u|\theta\,; \qquad \mathbf{v}_2 = 1|\phi\,. \tag{3.38}$$

At order $\mathcal{O}(k_{\mu}{}^{\frac{1}{2}})$, we find

$$\begin{aligned}\mathbf{\Delta}_{1}^{\mathbf{z}} = \frac{1}{2\pi\mathrm{i}}\Big\{ &-\frac{1}{2}\log k_1 - \pi\mathrm{i} - \log z + k_2{}^{\frac{1}{2}}(1-u)\Big(\frac{1}{u(1-z)}\theta\psi + \frac{1}{u-z}\psi\phi \\ &+ \frac{z+uz-2u}{u(u-z)(1-z)}\theta\phi\Big) - k_1{}^{\frac{1}{2}}k_2{}^{\frac{1}{2}}\frac{1-u}{uz}\left((1-z)\theta\psi + (u-z)\psi\phi\right)\Big\} + \mathcal{O}(k_{\mu})\end{aligned} \tag{3.39}$$

$$\begin{aligned}\mathbf{\Delta}_{2}^{\mathbf{z}} = \frac{1}{2\pi\mathrm{i}}\Big\{ &-\frac{1}{2}\log k_2 - \pi\mathrm{i} + \log\frac{1-z}{u-z} + \frac{1}{u-z}\theta\psi + \frac{1}{1-z}\psi\phi \\ &- k_1{}^{\frac{1}{2}}\frac{1}{uz}\Big((u-z)\theta\psi + z(1-u)\theta\phi + u(1-z)\psi\phi\Big) \\ &- k_1{}^{\frac{1}{2}}k_2{}^{\frac{1}{2}}\frac{(1-u)^2}{u}\Big(\frac{1}{u-z}\theta\psi + \frac{1}{1-z}\psi\phi\Big)\Big\} + \mathcal{O}(k_{\mu})\,.\end{aligned} \tag{3.40}$$

Exponentiating these, we get

$$\begin{aligned}\mathrm{e}^{2\pi\mathrm{i}\vec{\epsilon}\cdot\mathbf{\Delta}_{\mathbf{z}}} = \mathrm{e}^{-\mathrm{i}\pi(\epsilon_1+\epsilon_2)}k_1^{-\frac{\epsilon_1}{2}}k_2^{-\frac{\epsilon_2}{2}}z^{-\epsilon_1}\left(\frac{1-z}{u-z}\right)^{\epsilon_2}\times \\ \Big[1 - \epsilon_1\Big\{-k_2{}^{\frac{1}{2}}(1-u)\Big(\frac{1}{u(1-z)}\theta\psi + \frac{1}{u-z}\psi\phi + \Big(\frac{u}{z-u} - \frac{1}{u(1-z)}\Big)\theta\phi\Big) \\ - k_1{}^{\frac{1}{2}}k_2{}^{\frac{1}{2}}\frac{1-u}{uz}\Big((1-z)\theta\psi + (u-z)\psi\phi\Big)\Big\} \\ + \epsilon_2\Big\{\frac{1}{u-z}\theta\psi + \frac{1}{1-z}\psi\phi - k_1{}^{\frac{1}{2}}\frac{1}{uz}\Big((u-z)\theta\psi + z(1-u)\theta\phi \\ + u(1-z)\psi\phi\Big) - k_1{}^{\frac{1}{2}}k_2{}^{\frac{1}{2}}\frac{(1-u)^2}{u}\Big(\frac{1}{u-z}\theta\psi + \frac{1}{1-z}\psi\phi\Big)\Big\}\Big] + \mathcal{O}(k_{\mu})\,.\end{aligned} \tag{3.41}$$

The Prym differentials $\Omega_{\mu}^{\vec{\epsilon}}$ are holomorphic differentials; the natural analogues on SRSs are holomorphic volume forms: sections of the Berezinian bundle. Just as holomorphic differentials can be written locally as $\mathrm{d}z\,\partial_z f(z)$, sections of the Berezian can be written locally as $[\mathrm{d}z|\mathrm{d}\psi]\,D_{\psi}\,f(z|\psi)$, the combination being invariant under change of superconformal coordinates [60]. We note that we can write equation Eq. (3.35) for $\zeta_{\mu}^{\vec{\epsilon}}$ as

$$\begin{aligned}\zeta_{\mu}^{\epsilon}(z) = \mathrm{d}z\,\frac{\partial}{\partial z}\Big(&\sum_{\alpha}{}^{(\mu)}\mathrm{e}^{2\pi\mathrm{i}(\vec{\epsilon}\cdot N_{\alpha}+\epsilon_{\mu})}\log\Big[\frac{z-T_{\alpha}(\eta_{\mu})}{z-T_{\alpha}(\xi_{\mu})}\Big] \\ &+ (1-\mathrm{e}^{2\pi\mathrm{i}\epsilon_{\mu}})\sum_{\alpha}\mathrm{e}^{2\pi\mathrm{i}\vec{\epsilon}\cdot N_{\alpha}}\log\Big[\frac{z-T_{\alpha}(z_0)}{z-T_{\alpha}(\alpha_{\mu}^{\alpha})}\Big]\Big)\,,\end{aligned} \tag{3.42}$$

so to find the corresponding SRS volume forms we replace the expressions inside the logarithms with their natural superconformal analogues and replace $\mathrm{d}z\,\partial_z \mapsto [\mathrm{d}z|\mathrm{d}\psi]\,D_{\psi}$.

This yields

$$\boldsymbol{\zeta}^{\vec{\epsilon}}_{\mu}(z|\psi) = [\mathrm{d}z|\mathrm{d}\psi]\, D_\psi\Big(\sum_\alpha{}^{(\mu)} \mathrm{e}^{2\pi\mathrm{i}(\vec{\epsilon}\cdot N_\alpha+\epsilon_\mu)} \log\Big[\frac{\langle \mathbf{z}|\mathbf{T}_\alpha|\mathbf{u}_\mu\rangle}{\langle \mathbf{z}|\mathbf{T}_\alpha|\mathbf{v}_\mu\rangle}\Big] + (1-\mathrm{e}^{2\pi\mathrm{i}\epsilon_\mu})\sum_\alpha \mathrm{e}^{2\pi\mathrm{i}\vec{\epsilon}\cdot N_\alpha}\log\Big[\frac{\langle \mathbf{z}|\mathbf{T}_\alpha|\mathbf{z}_0\rangle}{\langle \mathbf{z}|\mathbf{T}_\alpha|\mathbf{a}_\mu\rangle}\Big]\Big) \tag{3.43}$$

$$= [\mathrm{d}z|\mathrm{d}\psi]\,\Big(\sum_\alpha{}^{(\mu)} \mathrm{e}^{2\pi\mathrm{i}(\vec{\epsilon}\cdot N_\alpha+\epsilon_\mu)}\Big[\frac{\langle \mathbf{z}|\Phi\mathbf{T}_\alpha|\mathbf{u}_\mu\rangle}{\langle \mathbf{z}|\mathbf{T}_\alpha|\mathbf{u}_\mu\rangle} - \frac{\langle \mathbf{z}|\Phi\mathbf{T}_\alpha|\mathbf{v}_\mu\rangle}{\langle \mathbf{z}|\mathbf{T}_\alpha|\mathbf{v}_\mu\rangle}\Big] + (1-\mathrm{e}^{2\pi\mathrm{i}\vec{\epsilon}_\mu})\sum_\alpha \mathrm{e}^{2\pi\mathrm{i}\epsilon\cdot N_\alpha}\Big[\frac{\langle \mathbf{z}|\Phi\mathbf{T}_\alpha|\mathbf{z}_0\rangle}{\langle \mathbf{z}|\mathbf{T}_\alpha|\mathbf{z}_0\rangle} - \frac{\langle \mathbf{z}|\Phi\mathbf{T}_\alpha|\mathbf{a}_\mu\rangle}{\langle \mathbf{z}|\mathbf{T}_\alpha|\mathbf{a}_\mu\rangle}\Big]\Big) \tag{3.44}$$

where we've used Φ defined in Eq. (2.239), $|\mathbf{z}_0\rangle$ is an arbitrary base point, and

$$|\mathbf{a}^\alpha_\mu\rangle = \begin{cases} |\mathbf{u}_\mu\rangle & \text{if } \mathbf{T}_\alpha = \mathbf{T}_\beta\mathbf{S}^\ell_\mu \text{ with } \ell\geq 1 \\ |\mathbf{v}_\mu\rangle & \text{otherwise.}\end{cases} \tag{3.45}$$

Then we can write down a basis of $(h-1)$ holomorphic volume forms $\boldsymbol{\Omega}^{\vec{\epsilon}}_\nu(z)$ with the expected monodromies along the homology cycles using the analogue of Eq. (3.34), noting that the dependence on the base point $|\mathbf{z}_0\rangle$ cancels out:

$$\boldsymbol{\Omega}^{\vec{\epsilon}}_\nu(z|\psi) = \boldsymbol{\zeta}^{\vec{\epsilon}}_\nu(z|\psi) - \frac{1-\mathrm{e}^{2\pi\mathrm{i}\epsilon_\nu}}{1-\mathrm{e}^{2\pi\mathrm{i}\epsilon_h}}\boldsymbol{\zeta}^{\vec{\epsilon}}_h(z|\psi) \qquad \nu = 1,\ldots,(h-1)\,. \tag{3.46}$$

We can calculate $\boldsymbol{\Omega}^{\vec{\epsilon}}_\nu(z|\psi)$ as a series expansion in $\sqrt{k_\mu}$. Truncating to finite order, we only need to sum Eq. (3.44) over finitely many terms of the super-Schottky group, because if the contribution from $\mathbf{T}_\alpha$ is $\mathcal{O}(k_\alpha{}^{\frac{1}{2}})$ and the left-most factor of $\mathbf{T}_\alpha$ is not $\mathbf{S}^{\pm1}_\mu$, then the contribution from $\mathbf{S}^{\pm\ell}_\mu\mathbf{T}_\alpha$ is $\mathcal{O}(k_\mu{}^{\frac{\ell}{2}}k_\alpha{}^{\frac{1}{2}})$. This means that if we only want to compute to $\mathcal{O}(\sqrt{k_\mu})$ for $h=2$ then we only need to sum over

$$\mathbf{T}_\alpha \in \{\mathrm{Id}, \mathbf{S}^{\pm1}_1, \mathbf{S}^{\pm1}_2, (\mathbf{S}_1\mathbf{S}_2)^{\pm1}, (\mathbf{S}^{-1}_1\mathbf{S}_2)^{\pm1}, (\mathbf{S}_1\mathbf{S}^{-1}_2)^{\pm1}, (\mathbf{S}_2\mathbf{S}_1)^{\pm1}\}\,. \tag{3.47}$$

We obtain, using the fixed points given in Eq. (4.24),

$$\begin{aligned}\boldsymbol{\Omega}^{\vec{\epsilon}}_1(z|\psi) = [\mathrm{d}z|\mathrm{d}\psi]\Big[&-\frac{(1-\mathcal{S}^{\vec{\epsilon}}_1)\mathcal{S}^{\vec{\epsilon}}_2\theta}{(1-\mathcal{S}^{\vec{\epsilon}}_2)(u-z)} + \frac{(1-\mathcal{S}^{\vec{\epsilon}}_1)\phi}{(1-\mathcal{S}^{\vec{\epsilon}}_2)(1-z)} \\ &+ \frac{\big(\mathcal{S}^{\vec{\epsilon}}_1 u(1-\mathcal{S}^{\vec{\epsilon}}_2) - (\mathcal{S}^{\vec{\epsilon}}_1 - \mathcal{S}^{\vec{\epsilon}}_2 - \mathcal{S}^{\vec{\epsilon}}_1\mathcal{S}^{\vec{\epsilon}}_2 u + u)z + z^2(1-\mathcal{S}^{\vec{\epsilon}}_2)\big)\,\psi}{(1-\mathcal{S}^{\vec{\epsilon}}_2)(u-z)(1-z)z} \\ &- k_1{}^{\frac{1}{2}}\Big\{-\frac{(1-\mathcal{S}^{\vec{\epsilon}}_1)\mathcal{S}^{\vec{\epsilon}}_1(\mathcal{S}^{\vec{\epsilon}}_2\theta-\phi)}{(1-\mathcal{S}^{\vec{\epsilon}}_2)z} + \frac{(1-\mathcal{S}^{\vec{\epsilon}}_1)(\mathcal{S}^{\vec{\epsilon}}_2\theta - u\phi)}{\mathcal{S}^{\vec{\epsilon}}_1(1-\mathcal{S}^{\vec{\epsilon}}_2)u}\Big\} \\ &- k_2{}^{\frac{1}{2}}\Big\{-\frac{\mathcal{S}^{\vec{\epsilon}}_2(1-\mathcal{S}^{\vec{\epsilon}}_1)\theta}{u-z} + \frac{\mathcal{S}^{\vec{\epsilon}}_2(1-\mathcal{S}^{\vec{\epsilon}}_1 u)\phi}{u-z} - \frac{(\mathcal{S}^{\vec{\epsilon}}_1-u)\theta}{\mathcal{S}^{\vec{\epsilon}}_2 u(1-z)} \\ &\qquad - \frac{\mathcal{S}^{\vec{\epsilon}}_2(1-\mathcal{S}^{\vec{\epsilon}}_1 u)\theta\phi\psi}{(u-z)^2} - \frac{(\mathcal{S}^{\vec{\epsilon}}_1-u)\theta\phi\psi}{\mathcal{S}^{\vec{\epsilon}}_2 u(1-z)^2} - \frac{(1-\mathcal{S}^{\vec{\epsilon}}_1)\phi}{\mathcal{S}^{\vec{\epsilon}}_2(1-z)}\Big\} \\ &+ k_1{}^{\frac{1}{2}}k_2{}^{\frac{1}{2}}\Big\{\frac{\mathcal{S}^{\vec{\epsilon}}_1\mathcal{S}^{\vec{\epsilon}}_2(\phi - \mathcal{S}^{\vec{\epsilon}}_1 u\phi - (1-\mathcal{S}^{\vec{\epsilon}}_1)\theta)}{z}\end{aligned} \tag{3.48}$$

$$
\begin{aligned}
&+\frac{\mathcal{S}_2^{\vec{\epsilon}}(\theta(1-\mathcal{S}_1^{\vec{\epsilon}})-\phi+\mathcal{S}_1^{\vec{\epsilon}}u\phi)}{\mathcal{S}_1^{\vec{\epsilon}}u}-\frac{((u-\mathcal{S}_1^{\vec{\epsilon}})\theta-(1-\mathcal{S}_1^{\vec{\epsilon}})u\phi)}{\mathcal{S}_1^{\vec{\epsilon}}\mathcal{S}_2^{\vec{\epsilon}}u}\\
&+\frac{\mathcal{S}_1^{\vec{\epsilon}}(u\theta-\mathcal{S}_1^{\vec{\epsilon}}\theta-u\phi(1-\mathcal{S}_1^{\vec{\epsilon}}))}{\mathcal{S}_2^{\vec{\epsilon}}uz}+\frac{(1-\mathcal{S}_1^{\vec{\epsilon}})\mathcal{S}_1^{\vec{\epsilon}}(1-u)\theta}{(1-\mathcal{S}_2^{\vec{\epsilon}})u(1-z)}\\
&-\frac{(1-\mathcal{S}_1^{\vec{\epsilon}})\mathcal{S}_1^{\vec{\epsilon}}(1-u)\phi}{(1-\mathcal{S}_2^{\vec{\epsilon}})\mathcal{S}_2^{\vec{\epsilon}}u(1-z)}-\frac{(1-\mathcal{S}_1^{\vec{\epsilon}})\mathcal{S}_1^{\vec{\epsilon}}\mathcal{S}_2^{\vec{\epsilon}}(1-u)(\mathcal{S}_2^{\vec{\epsilon}}\theta-\phi)}{(1-\mathcal{S}_2^{\vec{\epsilon}})(u-z)}\\
&-\frac{(1-\mathcal{S}_1^{\vec{\epsilon}})\mathcal{S}_1^{\vec{\epsilon}}\mathcal{S}_2^{\vec{\epsilon}}(1-u)\theta\phi\psi}{(1-\mathcal{S}_2^{\vec{\epsilon}})(u-z)^2}-\frac{(1-\mathcal{S}_1^{\vec{\epsilon}})^2(1+\mathcal{S}_1^{\vec{\epsilon}})(1-u)\theta\phi\psi}{\mathcal{S}_1^{\vec{\epsilon}}(1-\mathcal{S}_2^{\vec{\epsilon}})u(1-z)^2}\\
&+\frac{(1-\mathcal{S}_1^{\vec{\epsilon}})(1-u)(u\phi-\mathcal{S}_2^{\vec{\epsilon}}\theta)}{\mathcal{S}_1^{\vec{\epsilon}}(1-\mathcal{S}_2^{\vec{\epsilon}})\mathcal{S}_2^{\vec{\epsilon}}(1-z)u}+\frac{(1-\mathcal{S}_1^{\vec{\epsilon}})(\mathcal{S}_2^{\vec{\epsilon}})^2(1-u)\theta}{\mathcal{S}_1^{\vec{\epsilon}}(1-\mathcal{S}_2^{\vec{\epsilon}})u(u-z)}\\
&+\frac{(1-\mathcal{S}_1^{\vec{\epsilon}})\mathcal{S}_2^{\vec{\epsilon}}(1-u)\theta\phi\psi}{\mathcal{S}_1^{\vec{\epsilon}}(1-\mathcal{S}_2^{\vec{\epsilon}})(u-z)^2}-\frac{(1-\mathcal{S}_1^{\vec{\epsilon}})\mathcal{S}_2^{\vec{\epsilon}}(1-u)\phi}{\mathcal{S}_1^{\vec{\epsilon}}(1-\mathcal{S}_2^{\vec{\epsilon}})(u-z)}\Big\}\Big]+\mathcal{O}(k_\mu)\,,
\end{aligned}
$$

where

$$\mathcal{S}_\mu^{\vec{\epsilon}}=\mathrm{e}^{2\pi\mathrm{i}\epsilon_\mu}\,. \tag{3.49}$$

In our calculation of the twisted super-period matrix, the Prym differential appears not with the monodromies $\vec{\epsilon}$ but with $(\vec{\epsilon}\cdot\boldsymbol{\tau})$. All of the dependence of $\boldsymbol{\Omega}_1^{\vec{\epsilon}}$ on the monodromies enters only through the phases $\mathcal{S}_\mu^{\vec{\epsilon}}$, so we should replace $\mathcal{S}_\mu^{\vec{\epsilon}}$ in Eq. (3.48) with $\mathcal{S}_\mu^{\vec{\epsilon}\cdot\boldsymbol{\tau}}=\mathrm{e}^{2\pi\mathrm{i}(\vec{\epsilon}\cdot\boldsymbol{\tau})_\mu}$. Using $\boldsymbol{\tau}$ from Eq. (2.242), we find

$$\mathcal{S}_1^{\vec{\epsilon}\cdot\boldsymbol{\tau}}=\mathrm{e}^{2\pi\mathrm{i}(\vec{\epsilon}\cdot\boldsymbol{\tau})_1}=k_1^{\epsilon_1}u^{\epsilon_2}\left(1-2\epsilon_1k_2{}^{\frac{1}{2}}(1-1/u)\theta\phi\right)+\mathcal{O}(k_\mu)\,, \tag{3.50}$$

$$\mathcal{S}_2^{\vec{\epsilon}\cdot\boldsymbol{\tau}}=\mathrm{e}^{2\pi\mathrm{i}(\vec{\epsilon}\cdot\boldsymbol{\tau})_2}=k_2^{\epsilon_2}u^{\epsilon_1}\left(1-2\epsilon_2k_1{}^{\frac{1}{2}}(1-1/u)\theta\phi\right)+\mathcal{O}(k_\mu)\,. \tag{3.51}$$

We now have the ingredients to compute $\boldsymbol{\tau}_\epsilon$. We write

$$\boldsymbol{\tau}_{\vec{\epsilon}}=\left\{\frac{\mathrm{e}^{-\mathrm{i}\pi(\epsilon_1+\epsilon_2)}}{\sin(\pi(\epsilon_1+\epsilon_2))}\int_{0|0}^{u|\theta}\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{\Delta}^z\cdot\vec{\epsilon}}(1-\mathcal{S}_2^{\vec{\epsilon}\cdot\boldsymbol{\tau}})\,\boldsymbol{\Omega}_1^{\vec{\epsilon}\cdot\tau}(z)\right\}+(\epsilon_\mu\to-\epsilon_\mu)\,. \tag{3.52}$$

We insert $\boldsymbol{\Omega}_1^{\vec{\epsilon}\cdot\tau}(z)$ from Eq. (3.48) and $\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{\Delta}^z\cdot\vec{\epsilon}}$ from Eq. (3.41) and carry out the integration over ψ, which amounts to selecting the coefficients of ψ; the boundary terms don't contribute. We get a sum of integrals of the form

$$
\begin{aligned}
\boldsymbol{\tau}_{\vec{\epsilon}}=\left\{\sum_I f_I(u,\mathcal{S}_\mu^{\vec{\epsilon}\cdot\boldsymbol{\tau}},k_\mu,\epsilon_\mu|\theta,\phi)\frac{\mathrm{e}^{-\mathrm{i}\pi(\epsilon_1+\epsilon_2)}}{\sin(\pi(\epsilon_1+\epsilon_2))}\int_0^u\mathrm{d}z\,z^{n_1^I-\epsilon_1}(1-z)^{n_2^I+\epsilon_2}(u-z)^{n_3^I-\epsilon_2}\right\}\\
+(\epsilon_\mu\to-\epsilon_\mu)\,.
\end{aligned}
\tag{3.53}
$$

We can evaluate these integral with the substitution $z=tu$, getting

$$\int_0^u\mathrm{d}z\;\frac{z^{n_1-\epsilon_1}(1-z)^{n_2+\epsilon_2}}{(u-z)^{-n_3+\epsilon_2}}=u^{1+n_1+n_3-\epsilon_1-\epsilon_3}\int_0^1\mathrm{d}t\,\frac{t^{n_1-\epsilon_1}(1-ut)^{n_2+\epsilon_2}}{(1-t)^{-n_3+\epsilon_2}} \tag{3.54}$$

$$
\begin{aligned}
&=u^{1+n_1+n_3-\epsilon_1-\epsilon_3}\frac{\Gamma(1+n_1-\epsilon_1)\Gamma(1+n_3-\epsilon_2)}{\Gamma(2+n_1+n_3-\epsilon_1-\epsilon_2)}\\
&\quad\times{}_2F_1(-n_2-\epsilon_2,1+n_1-\epsilon_1;2+n_1+n_3-\epsilon_1-\epsilon_2;u)
\end{aligned}
\tag{3.55}
$$

where the hypergeometric function is given by the integral representation

$$
{}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,. \qquad (3.56)
$$

In fact we can combine the factor of $1/\sin(\pi(\epsilon_1+\epsilon_2))$ in Eq. (3.53) with the Γ functions in Eq. (3.55) at this stage using the identity $\Gamma(x)\Gamma(1-x)\sin(\pi x) \equiv \pi$ to obtain

$$
\begin{aligned}
\frac{1}{\sin(\pi(\epsilon_1+\epsilon_2))}\frac{1}{\Gamma(2+n_1+n_3-\epsilon_1-\epsilon_2)} &= \frac{1}{\pi}(-1)^{-1-n_1-n_3} \\
&\quad \times \Gamma(-1-n_1-n_3+\epsilon_1+\epsilon_2) \qquad (3.57)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\frac{\int_0^u dz\, \frac{z^{n_1-\epsilon_1}(1-z)^{n_2+\epsilon_2}}{(u-z)^{-n_3+\epsilon_2}}}{\sin(\pi(\epsilon_1+\epsilon_2))} &= \frac{1}{\pi}(-1)^{-1-n_1-n_3}u^{1+n_1+n_3-\epsilon_1-\epsilon_3}\Gamma(1+n_1-\epsilon_1) \qquad (3.58)\\
&\quad \times \Gamma(1+n_3-\epsilon_2)\Gamma(-1-n_1-n_3+\epsilon_1+\epsilon_2) \\
&\quad \times {}_2F_1(-n_2-\epsilon_2, 1+n_1-\epsilon_1; 2+n_1+n_3-\epsilon_1-\epsilon_2; u)\,.
\end{aligned}
$$

This integration can be carried out for each term in Eq. (3.53). All calculations in this section can be carried out using symbolic computation on MATHEMATICA;[1] we arrive at the following expression for the determinant of the Prym period matrix:

$$
\begin{aligned}
\boldsymbol{\tau}_{\vec{\epsilon}} = {} & \frac{1}{4\pi^2} k_2^{-\frac{\epsilon_2}{2}} k_1^{-\frac{\epsilon_1}{2}} u^{-\epsilon_1-\epsilon_2}\Gamma(-\epsilon_1)\Gamma(-\epsilon_2)\Gamma(\epsilon_1+\epsilon_2) \\
& \times \Big(\epsilon_2\, {}_2F_1(-\epsilon_1, 1-\epsilon_2; -\epsilon_1-\epsilon_2+1; u)\,(k_1^{\epsilon_1}u^{\epsilon_2}-1)\,(k_2^{\epsilon_2}u^{\epsilon_1}-1) \\
& + (u-1)\, {}_2F_1(1-\epsilon_1, 1-\epsilon_2; -\epsilon_1-\epsilon_2+1; u)\,(\epsilon_2 - k_2^{\epsilon_2}u^{\epsilon_1}(\epsilon_1(k_1^{\epsilon_1}u^{\epsilon_2}-1)+\epsilon_2))\Big) \\
& + \frac{1}{4\pi^2}\epsilon_1\epsilon_2 u^{\frac{1}{2}(-\epsilon_1-\epsilon_2)}\Gamma(-\epsilon_1)\Gamma(-\epsilon_2)\Gamma(\epsilon_1+\epsilon_2)\ {}_2F_1(1-\epsilon_1, 1-\epsilon_2; -\epsilon_1-\epsilon_2+1; u) \\
& \times \big(k_1^{-\frac{\epsilon_1}{2}}u^{-\frac{\epsilon_2}{2}} - k_1^{\frac{\epsilon_1}{2}}u^{\frac{\epsilon_2}{2}}\big)\big(k_2^{-\frac{\epsilon_2}{2}}u^{-\frac{\epsilon_1}{2}} - k_2^{\frac{\epsilon_2}{2}}u^{\frac{\epsilon_1}{2}}\big)\theta\phi \\
& + \frac{k_1{}^{\frac{1}{2}}}{4\pi^2} k_1^{-\frac{3\epsilon_1}{2}} k_2^{-\frac{\epsilon_2}{2}} u^{-\epsilon_1-2\epsilon_2}\Gamma(1-\epsilon_2)\Big\{k_1^{\epsilon_1}(-u^{\epsilon_2})\Gamma(-\epsilon_1)\Gamma(\epsilon_1+\epsilon_2) \\
& \quad \times (\epsilon_1\, {}_2F_1(1-\epsilon_1, 1-\epsilon_2; -\epsilon_1-\epsilon_2+1; u) + \epsilon_2\, {}_2F_1(-\epsilon_1, 1-\epsilon_2; -\epsilon_1-\epsilon_2+1; u)) \\
& \quad \times \big(-k_2^{\epsilon_2}u^{\epsilon_1-1} - k_1^{2\epsilon_1}u^{2\epsilon_2-1} + k_1^{2\epsilon_1}k_2^{\epsilon_2}u^{\epsilon_1+2\epsilon_2}+1\big) \\
& \quad - \Gamma(2-\epsilon_1)\Gamma(\epsilon_1+\epsilon_2-1)\ {}_2F_1(2-\epsilon_1, 1-\epsilon_2; -\epsilon_1-\epsilon_2+2; u) \\
& \quad \times \Big(-k_1^{2\epsilon_1}u^{2\epsilon_2}\big(1-k_2^{\epsilon_2}u^{\epsilon_1+1}\big) - k_2^{\epsilon_2}u^{\epsilon_1} + u\Big) \\
& \quad + \Gamma(1-\epsilon_1)\Gamma(\epsilon_1+\epsilon_2)\ {}_2F_1(1-\epsilon_1, 1-\epsilon_2; -\epsilon_1-\epsilon_2+1; u) \\
& \quad \times \Big(-k_1^{\epsilon_1}u^{\epsilon_2+1}\big(k_2^{\epsilon_2}k_1^{\epsilon_1}u^{\epsilon_1+\epsilon_2}+1\big) + k_1^{\epsilon_1}u^{\epsilon_2-1}(k_2^{\epsilon_2}u^{\epsilon_1} + k_1^{\epsilon_1}u^{\epsilon_2}) \\
& \quad - \Big(1-k_1^{3\epsilon_1}u^{3\epsilon_2}\Big)(1-k_2^{\epsilon_2}u^{\epsilon_1})\Big)\Big\}\theta\phi \\
& + \frac{k_2{}^{\frac{1}{2}}}{4\pi^2}\Big\{-u^{-2\epsilon_1-\epsilon_2}\Gamma(1-\epsilon_1)\Gamma(-\epsilon_2-1)\,k_2^{-\frac{3\epsilon_2}{2}}
\end{aligned}
$$

[1]The MATHEMATICA notebook should be available at `http://www.samplayle.com/mathematica/superschottky.nb` or is otherwise obtainable from the author by email.

$$
\begin{aligned}
&\times \Big(\Gamma(\epsilon_1+\epsilon_2+1)\ {}_2F_1(1-\epsilon_1, 2-\epsilon_2; -\epsilon_1-\epsilon_2+1; u)\\
&- \Gamma(\epsilon_1+\epsilon_2)\big({}_2F_1(2-\epsilon_1, 2-\epsilon_2; -\epsilon_1-\epsilon_2+1; u)\\
&- {}_2F_1(1-\epsilon_1, 2-\epsilon_2; -\epsilon_1-\epsilon_2+1; u)\big)(1-\epsilon_1)\Big)\\
&\times \Big(k_2^{\epsilon_2}\Big(k_2^{\epsilon_2}(\epsilon_1-\epsilon_2-1)u^{\epsilon_1}+k_2^{2\epsilon_2}(\epsilon_2+1)u^{2\epsilon_1}+k_1^{\epsilon_1}\epsilon_1 u^{\epsilon_2}\Big)u^{\epsilon_1-1}\\
&- k_2^{\epsilon_2}\Big(k_2^{\epsilon_2}(\epsilon_1-\epsilon_2)u^{\epsilon_1}+k_2^{2\epsilon_2}\epsilon_2 u^{2\epsilon_1}+\epsilon_1+\epsilon_2\Big)u^{\epsilon_1}\\
&- k_1^{\epsilon_1}(k_2^{\epsilon_2}u^{\epsilon_1}+1)(k_2^{\epsilon_2}(k_2^{\epsilon_2}u^{\epsilon_1}+\epsilon_1-2)u^{\epsilon_1}+1)u^{\epsilon_2}\\
&+ \Big(k_2^{\epsilon_2}\big(u^{\epsilon_1+\epsilon_2}k_2^{\epsilon_2}\epsilon_1 k_1^{\epsilon_1}+\epsilon_1+\epsilon_2-1\big)u^{\epsilon_1}-\epsilon_2+1\big)u+\epsilon_2\Big)k_1^{-\frac{\epsilon_1}{2}}\\
&- u^{-2\epsilon_1-\epsilon_2+1}\Gamma(3-\epsilon_1)\Gamma(-\epsilon_2-1)\Gamma(\epsilon_1+\epsilon_2-1)\\
&\times\ {}_2F_1(3-\epsilon_1, 2-\epsilon_2; -\epsilon_1-\epsilon_2+2; u)\, k_2^{-\frac{3\epsilon_2}{2}}\\
&\times \Big(-k_2^{\epsilon_2}((1-u)\epsilon_1+1)u^{\epsilon_1}-k_2^{2\epsilon_2}\big(1-\big(u^{-1}-1\big)\epsilon_1\big)u^{2\epsilon_1}+k_2^{3\epsilon_2}u^{3\epsilon_1}\\
&+ k_1^{\epsilon_1}\Big(k_2^{\epsilon_2}(-u\epsilon_1+\epsilon_1-(1-u)\epsilon_2+1)u^{\epsilon_1-1}\\
&+ k_2^{2\epsilon_2}(-\epsilon_1-\epsilon_2+u(\epsilon_1+\epsilon_2+1))u^{2\epsilon_1}\\
&- k_2^{3\epsilon_2}(u-(1-u)\epsilon_2)u^{3\epsilon_1}-\epsilon_2+\frac{\epsilon_2-1}{u}\Big)u^{\epsilon_2}+1\Big)k_1^{-\frac{\epsilon_1}{2}}\\
&+ u^{-2\epsilon_1-\epsilon_2}\Gamma(2-\epsilon_1)\Gamma(-\epsilon_2-1)\Gamma(\epsilon_1+\epsilon_2)\ {}_2F_1(2-\epsilon_1, 2-\epsilon_2; -\epsilon_1-\epsilon_2+1; u)\\
&\times k_2^{-\frac{3\epsilon_2}{2}}\Big(k_2^{\epsilon_2}(\epsilon_1+u(-u\epsilon_1-\epsilon_2+2)+\epsilon_2)u^{\epsilon_1}\\
&+ k_2^{2\epsilon_2}\Big(-\frac{\epsilon_1}{u}+u(\epsilon_1-\epsilon_2)+\epsilon_2+2\Big)u^{2\epsilon_1}-k_2^{3\epsilon_2}((1-u)\epsilon_2+2)u^{3\epsilon_1}\\
&+ k_1^{\epsilon_1}\big(k_2^{\epsilon_2}\Big(u(\epsilon_1-\epsilon_2)+\epsilon_2-2-\frac{\epsilon_1}{u}\Big)u^{\epsilon_1}-k_2^{2\epsilon_2}(u(u\epsilon_1+\epsilon_2+2)-\epsilon_1-\epsilon_2)u^{2\epsilon_1}\\
&+ k_2^{3\epsilon_2}(u(\epsilon_2+2)-\epsilon_2)u^{3\epsilon_1}-(1-u)\epsilon_2+2\big)u^{\epsilon_2}+(\epsilon_2-2)u-\epsilon_2\Big)k_1^{-\frac{\epsilon_1}{2}}\Big\}\theta\phi\\
+&\ \frac{k_1{}^{\frac{1}{2}}k_2{}^{\frac{1}{2}}}{4\pi^2}(u-1)k_1^{-\frac{3\epsilon_1}{2}}k_2^{-\frac{3\epsilon_2}{2}}\Big\{2\Gamma(-\epsilon_1)\Gamma(-\epsilon_2-1)\Gamma(\epsilon_1+\epsilon_2+2)\\
&\times\ {}_2F_1(-\epsilon_1, 1-\epsilon_2; -\epsilon_1-\epsilon_2-1; u)\, k_1^{2\epsilon_1}\Big(u^{2\epsilon_1}k_2^{2\epsilon_2}-1\Big)\epsilon_2 u^{-2\epsilon_1-1}\\
&+ 2\Gamma(-\epsilon_1)\Gamma(-\epsilon_2)\Gamma(\epsilon_1+\epsilon_2)k_1^{2\epsilon_1}k_2^{3\epsilon_2}\epsilon_2\Big({}_2F_1(1-\epsilon_1, 1-\epsilon_2; -\epsilon_1-\epsilon_2+1; u)\epsilon_1\\
&+ {}_2F_1(-\epsilon_1, 1-\epsilon_2; -\epsilon_1-\epsilon_2+1; u)\epsilon_2\Big)u^{\epsilon_1-1}-\Gamma(2-\epsilon_1)\Gamma(-\epsilon_2-1)\Gamma(\epsilon_1+\epsilon_2)\\
&\times\ {}_2F_1(2-\epsilon_1, -\epsilon_2; -\epsilon_1-\epsilon_2+1; u)\Big(u^{2\epsilon_2}k_1^{2\epsilon_1}-1\Big)k_2^{2\epsilon_2}\epsilon_1 u^{-2\epsilon_2-1}\\
&+ \Gamma(1-\epsilon_1)\Gamma(-\epsilon_2-1)\Gamma(\epsilon_1+\epsilon_2)k_2^{2\epsilon_2}\\
&\times \big({}_2F_1(2-\epsilon_1, -\epsilon_2; -\epsilon_1-\epsilon_2+1; u)(\epsilon_1-1)\\
&+ {}_2F_1\big(1-\epsilon_1, -\epsilon_2; -\epsilon_1-\epsilon_2+1; u\big)(\epsilon_2+1)\big)u^{-2\epsilon_2-1}\\
&- \Gamma(1-\epsilon_1)\Gamma(-\epsilon_2)\Gamma(\epsilon_1+\epsilon_2)\ {}_2F_1(1-\epsilon_1, 2-\epsilon_2; -\epsilon_1-\epsilon_2+1; u)\, k_2^{\epsilon_2}\\
&\times \Big(u^{\epsilon_2}\big(u^{2\epsilon_2}\epsilon_2 k_1^{2\epsilon_1}+\big(u^{2\epsilon_2}k_1^{2\epsilon_1}-1\big)\epsilon_1+\epsilon_2\big)k_1^{\epsilon_1}+u\big(-u^{\epsilon_2}(\epsilon_1+\epsilon_2-1)k_1^{\epsilon_1}\\
&+ u^{2\epsilon_2}(\epsilon_1+2\epsilon_2+1)k_1^{2\epsilon_1}+u^{3\epsilon_2}(\epsilon_1+\epsilon_2-1)k_1^{3\epsilon_1}-\epsilon_1+2\epsilon_2-1\big)\Big)u^{-\epsilon_1-2\epsilon_2-1}\\
&- \Gamma(2-\epsilon_1)\Gamma(-\epsilon_2)\Gamma(\epsilon_1+\epsilon_2-1)\ {}_2F_1(2-\epsilon_1, 2-\epsilon_2; -\epsilon_1-\epsilon_2+2; u)\, k_2^{\epsilon_2}\\
&\times \Big(u^{\epsilon_2}(-\epsilon_1+2\epsilon_2+1)k_1^{\epsilon_1}+u^{2\epsilon_2}(u\epsilon_1+\epsilon_1+u\epsilon_2-\epsilon_2+1)k_1^{2\epsilon_1}
\end{aligned}
$$

$$
\begin{aligned}
&+ u^{3\epsilon_2}\left(\epsilon_1+2\epsilon_2-1\right)k_1^{3\epsilon_1}-(u+1)\epsilon_1+u\epsilon_2+\epsilon_2-1\Big)u^{-\epsilon_1-2\epsilon_2}\\
&-\Gamma\left(3-\epsilon_1\right)\Gamma\left(-\epsilon_2\right)\Gamma\left(\epsilon_1+\epsilon_2-2\right)\,{}_2F_1\left(3-\epsilon_1,2-\epsilon_2;-\epsilon_1-\epsilon_2+3;u\right)k_2^{\epsilon_2}\\
&\times\big(2u^{\epsilon_2}\epsilon_2k_1^{\epsilon_1}+\big(u^{2\epsilon_2}k_1^{2\epsilon_1}-1\big)\epsilon_1\big)u^{-\epsilon_1-2\epsilon_2+1}\\
&+2\Gamma\left(1-\epsilon_1\right)\Gamma\left(-\epsilon_2\right)\Gamma\left(\epsilon_1+\epsilon_2\right)\,{}_2F_1\left(1-\epsilon_1,1-\epsilon_2;-\epsilon_1-\epsilon_2+1;u\right)\\
&\times\big(u^{3\epsilon_2+1}k_1^{3\epsilon_1}+1\big)k_2^{3\epsilon_2}\epsilon_2u^{\epsilon_1-2\epsilon_2-1}-\Gamma\left(-\epsilon_1\right)\Gamma\left(-\epsilon_2-1\right)\Gamma\left(\epsilon_1+\epsilon_2+2\right)\\
&\times\,{}_2F_1\left(-\epsilon_1,-\epsilon_2;-\epsilon_1-\epsilon_2-1;u\right)k_1^{\epsilon_1}\left(u^{2\epsilon_2}k_1^{2\epsilon_1}-1\right)k_2^{2\epsilon_2}\epsilon_1u^{-\epsilon_2-1}\\
&+2\Gamma\left(3-\epsilon_1\right)\Gamma\left(-\epsilon_2-1\right)\Gamma\left(\epsilon_1+\epsilon_2-1\right)\,{}_2F_1\left(3-\epsilon_1,1-\epsilon_2;-\epsilon_1-\epsilon_2+2;u\right)k_1^{\epsilon_1}\\
&\times\left(u^{2\epsilon_1}k_2^{2\epsilon_2}-1\right)\epsilon_2u^{-2\epsilon_1-\epsilon_2}-\Gamma\left(-\epsilon_1\right)\Gamma\left(-\epsilon_2\right)\Gamma\left(\epsilon_1+\epsilon_2\right)\\
&\times k_1^{\epsilon_1}k_2^{\epsilon_2}\big(\,{}_2F_1\left(1-\epsilon_1,2-\epsilon_2;-\epsilon_1-\epsilon_2+1;u\right)\epsilon_1\\
&+\,{}_2F_1\left(-\epsilon_1,2-\epsilon_2;-\epsilon_1-\epsilon_2+1;u\right)\epsilon_2\big)\big(2u^{\epsilon_2}\epsilon_2k_1^{\epsilon_1}+\big(u^{2\epsilon_2}k_1^{2\epsilon_1}-1\big)\epsilon_1\big)u^{-\epsilon_1-\epsilon_2-1}\\
&+2\Gamma\left(2-\epsilon_1\right)\Gamma\left(-\epsilon_2\right)\Gamma\left(\epsilon_1+\epsilon_2-1\right)\\
&\times\,{}_2F_1\left(2-\epsilon_1,1-\epsilon_2;-\epsilon_1-\epsilon_2+2;u\right)k_1^{\epsilon_1}k_2^{3\epsilon_2}\epsilon_2u^{\epsilon_1-\epsilon_2}\\
&+\Gamma\left(2-\epsilon_1\right)\Gamma\left(-\epsilon_2-1\right)\Gamma\left(\epsilon_1+\epsilon_2\right)\,{}_2F_1\left(2-\epsilon_1,1-\epsilon_2;-\epsilon_1-\epsilon_2+1;u\right)\\
&\times\big(u^{\epsilon_2+1}\big(u^{2\epsilon_1}k_2^{2\epsilon_2}\big(\epsilon_1+2\epsilon_2-1\big)-2\epsilon_2\big)k_1^{\epsilon_1}\\
&+u^{2(\epsilon_1+\epsilon_2)}k_2^{2\epsilon_2}\left(\epsilon_2-u\left(\epsilon_1+\epsilon_2+1\right)\right)k_1^{2\epsilon_1}\\
&+u^{3\epsilon_2}\big(u^{2\epsilon_1+1}k_2^{2\epsilon_2}\left(-\epsilon_1+2\epsilon_2+1\right)-2\epsilon_2\big)k_1^{3\epsilon_1}\\
&-2u\epsilon_2+u^{2\epsilon_1}k_2^{2\epsilon_2}\left(\epsilon_2+u\left(\epsilon_1+\epsilon_2\right)\right)\big)u^{-2(\epsilon_1+\epsilon_2)-1}\\
&+\Gamma\left(1-\epsilon_1\right)\Gamma\left(-\epsilon_2-1\right)\Gamma\left(\epsilon_1+\epsilon_2\right)\big(\,{}_2F_1\left(2-\epsilon_1,1-\epsilon_2;-\epsilon_1-\epsilon_2+1;u\right)\left(\epsilon_1-1\right)\\
&+\,{}_2F_1\left(1-\epsilon_1,1-\epsilon_2;-\epsilon_1-\epsilon_2+1;u\right)\left(\epsilon_2+1\right)\big)\\
&\times\big(u^{2\epsilon_1+\epsilon_2}k_2^{2\epsilon_2}\left(\epsilon_1+(u-1)\epsilon_2-1\right)k_1^{\epsilon_1}-u^{2\epsilon_2}\big(k_2^{2\epsilon_2}\left(\epsilon_1-2\epsilon_2+1\right)u^{2\epsilon_1}+2\epsilon_2\big)k_1^{2\epsilon_1}\\
&+u^{3\epsilon_2}\big(u^{2\epsilon_1}k_2^{2\epsilon_2}\left(-\epsilon_1+u\epsilon_2+\epsilon_2+1\right)-2\epsilon_2\big)k_1^{3\epsilon_1}-2u\epsilon_2\\
&+u^{2\epsilon_1}k_2^{2\epsilon_2}\left(\epsilon_1+2\epsilon_2\right)\big)u^{-2(\epsilon_1+\epsilon_2)-1}\Big\}\theta\phi+\mathcal{O}(k_\mu)+\left(\epsilon_\mu\leftrightarrow-\epsilon_\mu\right). \qquad (3.59)
\end{aligned}
$$

Chapter 4

The Quantum Field Theory limit of String Diagrams

It has been known since the early days of string theory that in the zero-slope limit $\alpha' \to 0$, string theory can be described by an effective field theory in the target spacetime of Yang-Mills gauge theory coupled to general relativity [2, 3]. In fact the correspondence can be made even more precise: in the Schottky group the $\alpha' \to 0$ limit of a string theory amplitude is expressed as a sum of finitely many integrals, seemingly associated to the different types of states that can propagate through degenerating homology cycles (*i.e.* to the different sectors of the worldsheet CFT). In this section we perform the necessary calculations to show that these integrals are associated to the various Feynman diagrams on the QFT side of the correspondence.

4.1 The string theory setup

We consider a stack of N parallel $\mathrm{D}_{(d-1)}$-branes, spatially separated from each other in the directions perpendicular to their worldvolumes (Dirichlet-branes or 'D-branes' were mentioned first in [95]; it is shown that they are innate to type II theories in [96]). This breaks the symmetry of the worldvolume theory from $\mathrm{U}(N)$ to $\mathrm{U}(1)^N$ [97]; the theory is in the 'Coulomb phase'. Furthermore, each of the D-branes has a constant U(1) background field in the x_1–x_2 plane, whose field strength is equal to

$$F^i_{\mu\nu} = B^i\big(\eta_{\mu 1}\eta_{2\nu} - \eta_{\mu 2}\eta_{\nu 1}\big)\,. \tag{4.1}$$

When the D-branes are displaced from each other in the transverse directions (which amounts in the worldvolume theory to giving the massless scalar fields a vacuum expectation value (VEV) [97]) then the factor of the integration measure corresponding to the scalar sector of the spacetime theory must be modified by multiplication with a factor giving the contribution from the tension of strings stretched between two non-coincident D-branes. We are considering a system of N parallel $\mathrm{D}_{(d-1)}$-branes whose coordinates in the transverse directions are given by $(Y^i_I) = (Y^i_1, \ldots, Y^i_{N_\mathrm{s}})$ (see Fig. 4.2). A string

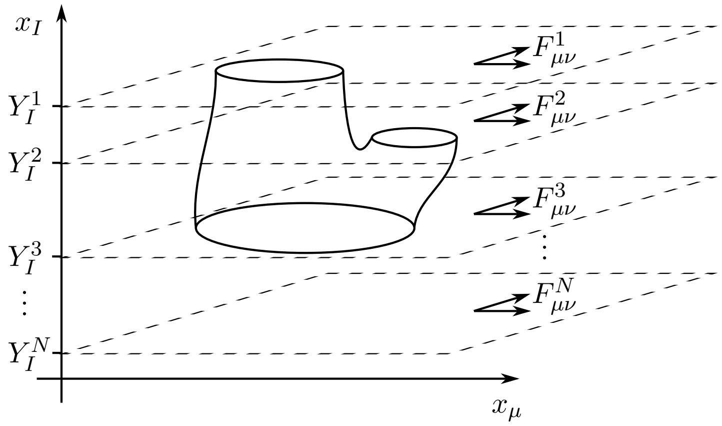


Figure 4.1: A stack of N parallel, separated $\mathrm{D}_{(d-1)}$-branes with background fields.

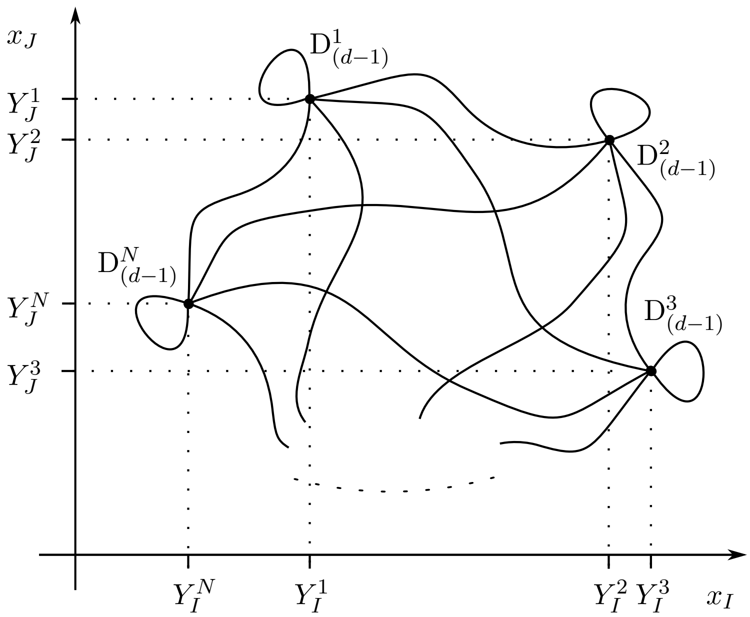


Figure 4.2: The positions of the N $\mathrm{D}_{(d-1)}$ branes in the x_I–x_J plane.

stretched between the ith and jth branes will have squared length

$$Y_{ji}^2 = \sum_{I=1}^{N_{\rm S}} (Y_I^i - Y_I^j)^2\,, \tag{4.2}$$

and will receive a classical contribution to its mass from the elastic potential energy associated with the stretching of the string,

$$m_{ij}^2 \;=\; (TY_{ij})^2 \;=\; \frac{Y_{ij}^2}{(2\pi\alpha')^2}\;, \tag{4.3}$$

where T is the fundamental string tension and α' is the related Regge-slope. Open strings that start and end on the same D-brane are uncharged and their mass is independent of Y_I^i.

The constant background magnetic fields on the D-brane worldvolumes manifest themselves in the worldsheet picture by altering the boundary conditions of the worldsheet CFTs. On the double cover of the surface, this gives twisted periodicities, *i.e.* non-trivial monodromies, to the zero modes in the two magnetized space directions, as described for bosonic strings in *e.g.* [84, 85, 86] and [87] and superstrings in [88, 89]; we follow the approach in section 2 of [92] and section 2 of [94]. This can be seen most readily in the path integral formalism.

4.2 The superstring partition function for the NS sector

The superstring vacuum amplitude for the NS sector of an h-loop open superstring with Neumann boundary conditions can be found in Ref. [9] in terms of super Schottky parameters, and may be written as

$$[\mathrm{d}\mathbf{m}]_h^0 \;=\; \left[\frac{1}{\mathrm{d}V_{\mathbf{v}_1\mathbf{u}_1\mathbf{v}_2}} \prod_{i=1}^{h} \frac{\mathrm{d}k_i}{k_i^{3/2}}\, \frac{\mathrm{d}\mathbf{u}_i\,\mathrm{d}\mathbf{v}_i}{\mathbf{v}_i \dot{-} \mathbf{u}_i}\right] \mathbf{F}_{\rm gh}(\mathbf{m})\, \mathbf{F}_{\rm gl}^{(0)}(\mathbf{m})\,. \tag{4.4}$$

In this expression, k_i, $\mathbf{u}_i \equiv u_i|\theta_i$ and $\mathbf{v}_i \equiv v_i|\phi_i$ are respectively the multiplier and the attractive and repulsive fixed super-points of the super Schottky group generator $\mathbf{S}_i$. The argument $\mathbf{m}$ in Eq. (4.4) denotes all of the moduli $k_i, u_i, v_i|\theta_i, \phi_i$, $i = 1, \ldots, h$.

In fact, there is an ambiguity in Eq. (4.4), because the half-integer power of k_i could indicate either of two branches of the function. The notation is to be understood in the following way: that $k_i^{\frac{1}{2}}$ indicates the smallest (in absolute value) of the eigenvalues of $\mathbf{S}_i$, and can therefore be either positive or negative. In fact, we will see later that the implementation of the GSO projection is equivalent to summing over all four possible pairs of choices of sign for $k_i^{\frac{1}{2}}$.

The notation $\mathbf{v}_i \dot{-} \mathbf{u}_i$ means the supersymmetric difference,

$$\mathbf{v}_i \dot{-} \mathbf{u}_i \equiv v_i - u_i + \theta_i\phi_i\,. \tag{4.5}$$

The factor

$$\frac{1}{\mathrm{d}V_{\mathbf{v}_1\mathbf{u}_1\mathbf{v}_2}} = \frac{\sqrt{(\mathbf{v}_1 \doteq \mathbf{u}_1)(\mathbf{u}_1 \doteq \mathbf{v}_2)(\mathbf{v}_2 \doteq \mathbf{v}_1)}}{\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{u}_1\mathrm{d}\mathbf{v}_2}\,\mathrm{d}\Theta_{\mathbf{v}_1\mathbf{u}_1\mathbf{v}_2} \tag{4.6}$$

takes into account the super-projective invariance of the integrand, which allows us to fix three bosonic and two fermionic variables. $\Theta_{\mathbf{v}_1\mathbf{u}_1\mathbf{v}_2}$ is the fermionic super-projective invariant of the three fixed points, which first appeared in ref. [39] and is given in Eq. (2.218). As a consequence, if we specialize to $h = 2$ loops then the factor in square brackets in Eq. (4.4) which we will call $\mathrm{d}\mu$ can also be written as

$$\mathrm{d}\mu \equiv \frac{1}{\mathrm{d}V_{\mathbf{v}_1\mathbf{u}_1\mathbf{v}_2}} \prod_{i=1}^{2} \frac{\mathrm{d}k_i}{k_i^{3/2}} \frac{\mathrm{d}\mathbf{u}_i\,\mathrm{d}\mathbf{v}_i}{\mathbf{v}_i \doteq \mathbf{u}_i} = \frac{\mathrm{d}k_1}{k_1^{3/2}} \frac{\mathrm{d}k_2}{k_2^{3/2}} \frac{\mathrm{d}\mathbf{u}_2\,\mathrm{d}\Theta_{\mathbf{v}_1\mathbf{u}_1\mathbf{v}_2}}{\mathbf{v}_2 \doteq \mathbf{u}_2} \sqrt{\frac{(\mathbf{u}_1 \doteq \mathbf{v}_2)(\mathbf{v}_2 \doteq \mathbf{v}_1)}{\mathbf{v}_1 \doteq \mathbf{u}_1}}\,. \tag{4.7}$$

As discussed in Ref. [98], it is important to specify the bosonic variables that are kept fixed when performing the Berezin integration over Grassmann variables. In the super Schottky parametrization, the objects entering in the NS vacuum energy, Eq. (4.4), can be expressed as [9]

$$\mathbf{F}_{\mathrm{gh}}(\mathbf{m}) = \frac{(1-k_1)^2\,(1-k_2)^2}{(1-k_1{}^{\frac{1}{2}})^2(1-k_2{}^{\frac{1}{2}})^2} \prod_\alpha{}' \prod_{n=2}^{\infty} \left(\frac{1-k_\alpha^n}{1-k_\alpha^{n-\frac{1}{2}}} \right)^2 , \tag{4.8}$$

$$\mathbf{F}_{\mathrm{gl}}^{(0)}(\mathbf{m}) = \Big[\det\left(\mathrm{Im}\,\boldsymbol{\tau}\right)\Big]^{-\frac{D}{2}} \prod_\alpha{}' \prod_{n=1}^{\infty} \left(\frac{1-k_\alpha^{n-\frac{1}{2}}}{1-k_\alpha^n} \right)^D , \tag{4.9}$$

$\boldsymbol{\tau}$ is the super-period matrix, discussed in section 2.3.3. The notation $\prod_\alpha'$ is defined after Eq. (2.248).

A multiplier k_α depends only on the supertrace of the OSp(1|2) matrix corresponding to $\mathbf{S}_\alpha$ (see Eq. (2.230) and section 2.3.2). Since a supertrace satisfies the same cyclic property as a trace, and a super-projective matrix and its inverse have the same multiplier, k_α is the same for every element in a primary class and so Eq. (4.8) and Eq. (4.9) are well-defined.

Note that as in Eq. (4.4), the multipliers k_α appear in this expression with half-integer powers. These are to be understood similarly, *i.e.* k_α denotes the smaller (in absolute value) of the two eigenvalues of $\mathbf{T}_\alpha$. k_α may be positive or negative. Each of the k_α's can be expressed as an algebraic function of the (super) moduli (including the $k_i{}^{\frac{1}{2}}$'s for $i = 1, 2$), and when we vary the signs of the $k_i{}^{\frac{1}{2}}$'s to carry out the GSO projection, the signs of the $k_\alpha{}^{\frac{1}{2}}$'s will also vary due to their dependence on $k_i{}^{\frac{1}{2}}$.

Since we want to compute vacuum amplitudes for open strings on parallel $\mathrm{D}_{(d-1)}$ branes, we need to alter Eq. (4.9) to reflect the Dirichlet boundary conditions in $N_\mathrm{s} \equiv D - d$ directions. This doesn't affect the orbital modes which enter Eq. (4.9) via the infinite product, but the factor of $[\det(\mathrm{Im}\,\boldsymbol{\tau})]^{-D/2}$ which arises from the integral over loop momenta should be modified by replacing $D \to d$. Anticipating that the factor from the N_s transverse directions will be the origin of scalar fields in the D-brane world-volume theory, we write

$$\mathbf{F}_{\mathrm{gl}}^{(0)}(\mathbf{m}) \to \mathbf{F}_{\mathrm{gl}}^{(0)}(\mathbf{m})\,\mathbf{F}_{\mathrm{scal}}^{(0)}(\mathbf{m})\,, \tag{4.10}$$

$$\mathbf{F}^{(0)}_{\rm gl}(\mathbf{m}) = \Big[\det\left(\mathrm{Im}\,\boldsymbol{\tau}\right)\Big]^{-\frac{d}{2}} \prod_\alpha{}' \prod_{n=1}^\infty \left(\frac{1-k_\alpha^{n-\frac{1}{2}}}{1-k_\alpha^n}\right)^d , \tag{4.11}$$

$$\mathbf{F}^{(0)}_{\rm scal}(\mathbf{m}) = \prod_\alpha{}' \prod_{n=1}^\infty \left(\frac{1-k_\alpha^{n-\frac{1}{2}}}{1-k_\alpha^n}\right)^{N_{\rm s}} . \tag{4.12}$$

In the presence of a constant background gauge field, the factor $\mathbf{F}^{(0)}_{\rm gl}$ in Eq. (4.11) gets modified, since string coordinates with Neumann boundary conditions propagate in space-time and are sensitive to such backgrounds. The relevant modification to the bosonic theory was derived in [16]. Using the same techniques, developed and described in [51, 16, 94], it is possible to generalize this construction to the Neveu-Schwarz spin structure of the RNS superstring. Switching on the background field amounts to multiplying $\mathbf{F}^{(0)}_{\rm gl}$ by a factor which depends on the twists $\vec{\epsilon} = (\epsilon_1, \epsilon_2)$ which are related to the strengths of the background fields on the three boundaries via Eq. (3.31). In terms of this, we have

$$\mathbf{F}^{(0)}_{\rm gl}(\mathbf{m}) \to \mathbf{F}^{(\vec{\epsilon})}_{\rm gl}(\mathbf{m}) = \mathcal{R}(\mathbf{m},\vec{\epsilon})\,\mathbf{F}^{(0)}_{\rm gl}(\mathbf{m}) , \tag{4.13}$$

where, assuming that the constant background field strength is non-zero in only one plane, we have

$$\begin{aligned}\mathcal{R}(\mathbf{m},\vec{\epsilon}) = \mathrm{e}^{-\mathrm{i}\pi\vec{\epsilon}\cdot\boldsymbol{\tau}\cdot\vec{\epsilon}}\,\frac{\det\left(\mathrm{Im}\,\boldsymbol{\tau}\right)}{\det\left(\mathrm{Im}\,\boldsymbol{\tau}_{\vec{\epsilon}}\right)} \prod_\alpha{}' \prod_{n=1}^\infty &\Bigg\{ \left(\frac{1-k_\alpha^{n-\frac{1}{2}}}{1-k_\alpha^n}\right)^{-2} \\ &\times \frac{\left(1-\mathrm{e}^{2\mathrm{i}\pi\vec{\epsilon}\cdot\boldsymbol{\tau}\cdot\vec{N}_\alpha}\,k_\alpha^{n-\frac{1}{2}}\right)\left(1-\mathrm{e}^{-2\mathrm{i}\pi\,\vec{\epsilon}\cdot\boldsymbol{\tau}\cdot\vec{N}_\alpha}\,k_\alpha^{n-\frac{1}{2}}\right)}{\left(1-\mathrm{e}^{2\pi\mathrm{i}\vec{\epsilon}\cdot\boldsymbol{\tau}\cdot\vec{N}_\alpha}\,k_\alpha^n\right)\left(1-\mathrm{e}^{-2\pi\mathrm{i}\vec{\epsilon}\cdot\boldsymbol{\tau}\cdot\vec{N}_\alpha}\,k_\alpha^n\right)}\Bigg\} .\end{aligned} \tag{4.14}$$

The h-component vector $\vec{N}_\alpha$ has integer-valued entries: the i-th entry counts how many times the generator $\mathbf{S}_i$ enters in the element of the super Schottky group $\mathbf{T}_\alpha$; we define $N^i_\alpha = 0$ for $\mathbf{T}_\alpha = \mathbf{Id}$ and $N^i_\alpha = N^i_\beta \pm 1$ for $\mathbf{T}_\alpha = \mathbf{S}_i^{\pm 1}\mathbf{T}_\beta$.

Inserting Eq. (4.14) in Eq. (4.13), we see that $\mathbf{F}^{(\vec{\epsilon})}_{\rm gl}$ can be factorized as the product of two terms

$$\mathbf{F}^{(\vec{\epsilon})}_{\rm gl}(\mathbf{m}) = \mathbf{F}_\perp(\mathbf{m})\,\mathbf{F}^{(\vec{\epsilon})}_\parallel(\mathbf{m}) \tag{4.15}$$

where $\mathbf{F}^{(\vec{\epsilon})}_\perp$ is equal to $\mathbf{F}^{(0)}_{\rm gl}$ with the replacement $d \to (d-2)$, and all dependence on the background field is contained in $\mathbf{F}^{(\vec{\epsilon})}_\parallel$ which reduces to $\mathbf{F}^{(0)}_{\rm gl}$ with the replacement $d \to 2$ in the limit $\vec{\epsilon} \to 0$. The notation is motivated in anticipation that in the QFT limit, $\mathbf{F}^{(\vec{\epsilon})}_\parallel$ will be the origin of gluons polarized in the plane of the background magnetic field, while $\mathbf{F}_\perp$ will be the origin of gluons polarized in the transverse directions. To be precise, we have

$$\mathbf{F}_\perp(\mathbf{m}) = \big[\det\left(\mathrm{Im}\,\boldsymbol{\tau}\right)\big]^{-\frac{d-2}{2}} \prod_\alpha{}' \prod_{n=1}^\infty \left(\frac{1-k_\alpha^{n-\frac{1}{2}}}{1-k_\alpha^n}\right)^{d-2} , \tag{4.16}$$

$$\mathbf{F}^{(\vec{\epsilon})}_\parallel(\mathbf{m}) = \frac{\mathrm{e}^{-\mathrm{i}\pi\vec{\epsilon}\cdot\boldsymbol{\tau}\cdot\vec{\epsilon}}}{\det\left(\mathrm{Im}\,\boldsymbol{\tau}_\epsilon\right)} \tag{4.17}$$

$$\times \prod_{\alpha}' \prod_{n=1}^{\infty} \frac{\left(1 - \mathrm{e}^{2\mathrm{i}\pi\vec{\epsilon}\cdot\boldsymbol{\tau}\cdot\vec{N}_\alpha}\, k_\alpha^{n-\frac{1}{2}}\right)\left(1 - \mathrm{e}^{-2\mathrm{i}\pi\vec{\epsilon}\cdot\boldsymbol{\tau})\cdot\vec{N}_\alpha}\, k_\alpha^{n-\frac{1}{2}}\right)}{\left(1 - \mathrm{e}^{2\pi\mathrm{i}\vec{\epsilon}\cdot\boldsymbol{\tau}\cdot\vec{N}_\alpha}\, k_\alpha^n\right)\left(1 - \mathrm{e}^{-2\pi\mathrm{i}\vec{\epsilon}\cdot\boldsymbol{\tau}\cdot\vec{N}_\alpha}\, k_\alpha^n\right)}\,.$$

When the D-branes are separated from each other in the transverse directions as in Fig. 4.2, $\mathbf{F}^{(0)}_{\mathrm{scal}}$ has to be multiplied by a factor accounting for the string tensions. The factor can be found by following the equivalent calculation for the bosonic theory in ref. [99] with the only essential difference that the period matrix τ is to be replaced with the super-period matrix $\boldsymbol{\tau}$. We find

$$\mathbf{F}^{(0)}_{\mathrm{scal}} \to \mathbf{F}^{(\vec{m}_I)}_{\mathrm{scal}} = \mathcal{Y}(\mathbf{m}, \vec{m}_I)\, \mathbf{F}^{(0)}_{\mathrm{scal}}\,; \tag{4.18}$$

$$\mathcal{Y}(\mathbf{m}, \vec{m}_I) \equiv \prod_{I=1}^{N_{\mathrm{s}}} \mathrm{e}^{2\pi\mathrm{i}\alpha' \vec{m}_I\cdot\boldsymbol{\tau}\cdot\vec{m}_I}\,. \tag{4.19}$$

Here $\vec{m}_I$ is a vector whose $h = 2$ components have dimensions of mass (but note that they can be negative) and encode the stretching in the x_I direction of strings propagating around the two open string handles. If we say the homology cycle a_1 separates boundaries on the ith and kth branes and the homology cycle a_2 separates boundaries on the kth and jth branes, then it is given by $\vec{m}_I = (m_I^{ij}, m_I^{jk})$, where

$$m_I^{ab} \equiv \frac{Y_I^a - Y_I^b}{2\pi\alpha'}\,. \tag{4.20}$$

As in Fig. 4.2, Y_I^a is the x_I co-ordinate of the D-brane labelled by a. Note that m_I^{ab} can be positive or negative depending on the order of a and b. If we use $m_I^{ij} = -m_I^{ki} - m_I^{jk}$, then we see that the factor depends only on the squares of the three m_I^{ab}'s, since

$$\vec{m}_I \cdot \boldsymbol{\tau} \cdot \vec{m}_I = (m_I^{ki})^2\, \boldsymbol{\tau}_{11} + (m_I^{jk})^2\, \boldsymbol{\tau}_{22} + \left((m_I^{ij})^2 - (m_I^{jk})^2 - (m_I^{ki})^2\right) \boldsymbol{\tau}_{12}\,. \tag{4.21}$$

It follows from Eq. (4.3) that the squared masses m_{ij}^2 of the stretched states are simply the sums of the squares of the m_I^{ij}'s. Therefore the product over the N_{s} transverse directions in Eq. (4.19) can be easily evaluated and we arrive at an expression for $\mathcal{Y}(\mathbf{m}, \vec{m}_I)$ in terms of the squared masses m_{ij}^2:

$$\mathcal{Y}(\mathbf{m}, \vec{m}_I) = \exp\left[2\pi\mathrm{i}\alpha'\Big(m_{ki}^2\boldsymbol{\tau}_{11} + m_{jk}^2\boldsymbol{\tau}_{22} + \left(m_{ij}^2 - m_{jk}^2 - m_{ki}^2\right)\boldsymbol{\tau}_{12}\Big)\right]. \tag{4.22}$$

In our setup, then, the integration measure in Eq. (4.4) is modified to

$$[\mathrm{d}\mathbf{m}]_2^{0,\vec{\epsilon},\vec{m}_I} = \mathrm{d}\mu\, \mathbf{F}_{\mathrm{gh}}(\mathbf{m})\, \mathbf{F}_{\parallel}^{(\vec{\epsilon})}(\mathbf{m})\, \mathbf{F}_{\perp}(\mathbf{m})\, \mathbf{F}^{\vec{m}_I}_{\mathrm{scal}}(\mathbf{m})\,, \tag{4.23}$$

where $\mathrm{d}\mu$ is defined in Eq. (4.7), $\mathbf{F}_{\mathrm{gh}}$ is defined in Eq. (4.8), $\mathbf{F}_{\parallel}^{(\vec{\epsilon})}$ is defined in Eq. (4.17), $\mathbf{F}_{\perp}$ is defined in Eq. (4.16) and $\mathbf{F}^{\vec{m}_I}_{\mathrm{scal}}$ is defined in Eq. (4.18).

In fact, all of the factors in Eq. (4.23) are invariant under super-projective transformations of the fixed super-points. We can use this to fix $3|2$ of them; a convenient gauge is

$$\mathbf{u}_1 = 0|0 \qquad \mathbf{v}_1 = \infty|0 \qquad \mathbf{u}_2 = u|\theta \qquad \mathbf{v}_2 = 1|\phi\,, \tag{4.24}$$

which leads to

$$\Theta_{\mathbf{v}_1\mathbf{u}_1\mathbf{v}_2} = \phi\,, \qquad \mathbf{v}_2 \dot{-} \mathbf{u}_2 = 1 - u + \theta\phi\,, \qquad \sqrt{\frac{(\mathbf{u}_1 \dot{-} \mathbf{v}_2)(\mathbf{v}_2 \dot{-} \mathbf{v}_1)}{\mathbf{v}_1 \dot{-} \mathbf{u}_1}} = 1\,. \tag{4.25}$$

Inserting these in Eq. (4.7), we can write the measure finally as

$$[\mathrm{d}\mathbf{m}]_2^{0,\vec{\epsilon},\vec{m}_I} = \frac{\mathrm{d}k_1}{k_1^{3/2}}\frac{\mathrm{d}k_2}{k_2^{3/2}}\frac{\mathrm{d}u}{y}\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathbf{F}_{\mathrm{gh}}(\mathbf{m})\,\mathbf{F}_{\|}^{(\vec{\epsilon})}(\mathbf{m})\,\mathbf{F}_{\perp}(\mathbf{m})\,\mathbf{F}_{\mathrm{scal}}^{\vec{m}_I}(\mathbf{m})\,, \tag{4.26}$$

where we've written

$$y \equiv \widehat{\Psi}_{\mathbf{u}_1\mathbf{v}_1\mathbf{u}_2\mathbf{v}_2} = 1 - u + \theta\phi\,, \tag{4.27}$$

which is the sewing parameter for the separating degeneration; the super-projective invariant cross ratio $\widehat{\Psi}_{\mathbf{z}_1\mathbf{z}_2\mathbf{z}_3\mathbf{z}_4}$ is defined in Eq. (2.219).

4.3 The field theory limit

4.3.1 Expanding in $k_i^{\frac{1}{2}}$

We are interested in computing the $\alpha' \to 0$ limit of the superstring amplitude. The measure includes a product over infinitely many primary classes in the super-Schottky group, with infinitely many powers of each; similarly the expression for super-period matrix includes a sum over infinitely many Schottky group elements. But all of the terms in the measure Eq. (4.26) can be expressed in terms of the 3|2 moduli of the SRS, and we can write them as series expansions in the multipliers $k_i^{\frac{1}{2}}$.

A term in the $k_i^{\frac{1}{2}}$ power series expansion of the measure has a natural interpretation as the term associated to string states of a particular excitation level propagating around the loops; a term proportional to $\mathrm{d}k_i\, k_i^{n/2}$ corresponds to the $(n+3)$th excited level. Therefore all terms with $n \geq -1$ get squared masses $m^2 = \frac{n+2}{2\alpha'}$ and become infinitely heavy in the limit $\alpha' \to 0$. Since $\mathrm{d}\mu$ in Eq. (4.7) includes factors of the form $\mathrm{d}k_i/k_i^{3/2}$, it is necessary to compute $\mathbf{F}_{\mathrm{gh}}(\mathbf{m})$, $\mathbf{F}_{\|}^{(\vec{\epsilon})}(\mathbf{m})$, $\mathbf{F}_{\perp}(\mathbf{m})$ and $\mathbf{F}_{\mathrm{scal}}^{\vec{m}_I}(\mathbf{m})$ only up to terms of order $k_i^{\frac{1}{2}}$ to get the full QFT amplitude.

This is aided by the fact that the multipliers of all but finitely many super-Schottky group elements vanish at order $k_1^{\frac{1}{2}}k_2^{\frac{1}{2}}$, so the infinite products and series reduce to finite sums. This is because the leading-order behaviour of the multiplier $k_\alpha = k(\mathbf{S}_\alpha)$ is related in a simple way to how many times the generators $\mathbf{S}_i$ and their inverses appear in the reduced word corresponding to $\mathbf{S}_\alpha$: we have

$$k\big(\mathbf{S}_i^{\pm 1}\mathbf{S}_\alpha\big)^{\frac{1}{2}} \in \mathcal{O}\big((k_i k_\alpha)^{\frac{1}{2}}\big) \tag{4.28}$$

if the left-most factor of $\mathbf{S}_\alpha$ is not $\mathbf{S}_i^{\mp 1}$. Thus, the square roots of multipliers of every super-Schottky group element not in the primary class of one of $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_1\mathbf{S}_2$, $\mathbf{S}_1^{-1}\mathbf{S}_2$ vanish at order ${k_1}^{\frac{1}{2}}{k_2}^{\frac{1}{2}}$.

We can compute expressions for all of the factors $\mathbf{F}_{\mathrm{gh}}(\mathbf{m})$, $\mathbf{F}_{\|}^{(\vec{\epsilon})}(\mathbf{m})$, $\mathbf{F}_{\perp}(\mathbf{m})$ and $\mathbf{F}_{\mathrm{scal}}^{\vec{m}_I}(\mathbf{m})$ at order ${k_1}^{\frac{1}{2}}{k_2}^{\frac{1}{2}}$. We will begin with $\mathbf{F}_{\mathrm{gh}}$ defined in Eq. (4.8). Most terms can be neglected: every term in the infinite product differs from unity only by terms of order at least $k_i^{3/2}$, and the numerator of the fraction by terms of order k_i. It becomes simply

$$\mathbf{F}_{\mathrm{gh}}(\mathbf{m}) = (1 + 2{k_1}^{\frac{1}{2}})(1 + 2{k_2}^{\frac{1}{2}}) + \mathcal{O}(k_\mu)\,. \tag{4.29}$$

Next we find $\mathbf{F}_\perp$, defined in Eq. (4.16). It is given by

$$\begin{aligned}\mathbf{F}_\perp = {}& \big[\det(\mathrm{Im}\,\boldsymbol{\tau})\big]^{-\frac{d-2}{2}}\,\big(1-(d-2){k_1}^{\frac{1}{2}}\big)\big(1-(d-2){k_2}^{\frac{1}{2}}\big) \\ & \times\Big(1-(d-2)k\big(\mathbf{S}_1^{-1}\mathbf{S}_2\big)^{\frac{1}{2}}\Big)\Big(1-(d-2)k\big(\mathbf{S}_1\mathbf{S}_2\big)^{\frac{1}{2}}\Big) + \mathcal{O}(k_\mu)\,,\end{aligned} \tag{4.30}$$

or if we expand the ${k_\alpha}^{\frac{1}{2}}$'s as in Eq. (2.231) and Eq. (2.233), we find

$$\begin{aligned}\mathbf{F}_\perp = {}& \big[\det(\mathrm{Im}\,\boldsymbol{\tau})\big]^{-\frac{d-2}{2}}\,\Big\{(1-(d-2){k_1}^{\frac{1}{2}})(1-(d-2){k_2}^{\frac{1}{2}}) \\ & + (d-2){k_1}^{\frac{1}{2}}{k_2}^{\frac{1}{2}}\,\frac{y}{u} - (d-2){k_1}^{\frac{1}{2}}{k_2}^{\frac{1}{2}}\,y\Big\} + \mathcal{O}(k_\mu)\,.\end{aligned} \tag{4.31}$$

The ${k_i}^{\frac{1}{2}}$-expansion of $\det(\mathrm{Im}\,\boldsymbol{\tau})$ is given in Eq. (2.243), which leads to

$$\begin{aligned}\big[\det(\mathrm{Im}\,\boldsymbol{\tau})\big]^{-\frac{d-2}{2}} = {}& \frac{(4\pi^2)^{\frac{d-2}{2}}}{\big(\log k_1 \log k_2 - (\log u)^2\big)^{\frac{d-2}{2}}} \\ & \times\,\Big(1-(d-2)\frac{y}{u}\frac{{k_1}^{\frac{1}{2}}\log k_1 + {k_2}^{\frac{1}{2}}\log k_2}{\log k_1 \log k_2 - (\log u)^2}\theta\phi\Big) + \mathcal{O}(k_\mu)\,.\end{aligned} \tag{4.32}$$

Next we should calculate $\mathbf{F}_{\|}^{(\vec{\epsilon})}$ from Eq. (4.17). The first factor can be found using $\boldsymbol{\tau}$ from Eq. (2.242) and is given by

$$\mathrm{e}^{-\mathrm{i}\pi\vec{\epsilon}\cdot\boldsymbol{\tau}\cdot\vec{\epsilon}} = k_1^{-\frac{\epsilon_1^2}{2}}k_2^{-\frac{\epsilon_2^2}{2}}u^{-\epsilon_1\epsilon_2}\Big(1-\frac{y}{u}\big({k_2}^{\frac{1}{2}}\epsilon_1^2 + k_1^{\frac{1}{2}}\epsilon_2^2\big)\theta\phi\Big) + \mathcal{O}(k_\mu)\,. \tag{4.33}$$

$\det(\mathrm{Im}\,\boldsymbol{\tau}_{\vec{\epsilon}})^{-1}$ can be found by inverting the right hand side of Eq. (3.59). The remaining factor is given by

$$\begin{aligned}\prod_\alpha{}'\prod_{n=1}^{\infty} & \frac{\big(1-\mathrm{e}^{2\mathrm{i}\pi\vec{\epsilon}\cdot\tau\cdot\vec{N}_\alpha}\,k_\alpha^{n-\frac{1}{2}}\big)\big(1-\mathrm{e}^{-2\mathrm{i}\pi\vec{\epsilon}\cdot\tau\cdot\vec{N}_\alpha}\,k_\alpha^{n-\frac{1}{2}}\big)}{\big(1-\mathrm{e}^{2\pi\mathrm{i}\vec{\epsilon}\cdot\tau\cdot\vec{N}_\alpha}\,k_\alpha^{n}\big)\big(1-\mathrm{e}^{-2\pi\mathrm{i}\vec{\epsilon}\cdot\tau\cdot\vec{N}_\alpha}\,k_\alpha^{n}\big)} \\ & = \Big(1-{k_1}^{\frac{1}{2}}\big(k_1^{\epsilon_1}u^{\epsilon_2} + k_1^{-\epsilon_1}u^{-\epsilon_2} \\ & \qquad - 2\epsilon_1\theta\phi\frac{y}{u}k_2^{\frac{1}{2}}\,(k_1^{\epsilon_1}u^{\epsilon_2} - k_1^{-\epsilon_1}u^{-\epsilon_2})\big)\Big) \\ & \quad\times\Big(1-{k_2}^{\frac{1}{2}}\big(k_2^{\epsilon_2}u^{\epsilon_1} + k_2^{-\epsilon_2}u^{-\epsilon_1}\end{aligned} \tag{4.34}$$

$$
\begin{aligned}
&\qquad - 2\epsilon_2\theta\phi\frac{y}{u}k_1^{\frac{1}{2}}(k_2^{\epsilon_2}u^{\epsilon_1} - k_2^{-\epsilon_2}u^{-\epsilon_1}))\Big)\\
&- k_1{}^{\frac{1}{2}}k_2{}^{\frac{1}{2}}(k_1^{\epsilon_1}k_2^{\epsilon_2}u^{\epsilon_1+\epsilon_2} + k_1^{-\epsilon_1}k_2^{-\epsilon_2}u^{-\epsilon_1-\epsilon_2})y\\
&\quad + k_1{}^{\frac{1}{2}}k_2{}^{\frac{1}{2}}\big(k_1^{\epsilon_1}k_2^{-\epsilon_2}u^{\epsilon_1-\epsilon_2} + k_1^{-\epsilon_1}k_2^{\epsilon_2}u^{\epsilon_2-\epsilon_1}\big)\frac{y}{u} + \mathcal{O}(k_\mu)\,.
\end{aligned}
$$

Lastly, we need to find an expression for $\mathbf{F}^{(\vec{m}_I)}_{\text{scal}}$, defined in Eq. (4.18). We can use Eq. (4.22) to expand $\mathcal{Y}$ in the $k_i{}^{\frac{1}{2}}$ using the expression for $\boldsymbol{\tau}$ in Eq. (2.242), we get

$$
\begin{aligned}
\mathcal{Y}(\mathbf{m}, \vec{m}_I) = {}& k_1^{\alpha' m_1^2}k_2^{\alpha' m_2^2}u^{\alpha'(m_3^2-m_1^2-m_2^2)}\\
&\times\Big(1 + 2\,\alpha'\frac{y}{u}k_1{}^{\frac{1}{2}}\,m_2^2 + k_2{}^{\frac{1}{2}}\,m_1^2)\Big) + \mathcal{O}(k_\mu)\,,
\end{aligned}
\tag{4.35}
$$

where we've renamed $(m_{ki}^2, m_{jk}^2, m_{ij}^2) \to (m_1^2, m_2^2, m_3^2)$. The remaining factor in $\mathbf{F}^{(\vec{m}_I)}_{\text{scal}}$ is given by

$$
\mathbf{F}^{(0)}_{\text{scal}}(\mathbf{m}) = (1 - N_{\text{s}}\,k_1{}^{\frac{1}{2}})(1 - N_{\text{s}}\,k_2{}^{\frac{1}{2}}) + N_{\text{s}}\,k_1{}^{\frac{1}{2}}k_2{}^{\frac{1}{2}}\,\frac{y}{u} - N_{\text{s}}\,k_1{}^{\frac{1}{2}}k_2{}^{\frac{1}{2}}\,y + \mathcal{O}(k_\mu)\,. \tag{4.36}
$$

We now have all of the factors of Eq. (4.26); they can be combined to give an expression for the full integration measure,

$$
\begin{aligned}
[\mathrm{d}\mathbf{m}]_2^0 = {}& \frac{\mathrm{d}k_1}{k_1^{3/2}}\,\frac{\mathrm{d}k_2}{k_2^{3/2}}\,\frac{\mathrm{d}u}{y}\,\mathrm{d}\theta\,\mathrm{d}\phi\,\frac{\big[\det(\mathrm{Im}\,\boldsymbol{\tau})\big]^{-\frac{d-2}{2}}}{\det(\mathrm{Im}\,\boldsymbol{\tau}_{\vec{\epsilon}})}\\
& k_1^{\alpha' m_1^2}k_2^{\alpha' m_2^2}u^{\alpha'(m_3^2-m_1^2-m_2^2)}k_1^{-\frac{\epsilon_1^2}{2}}k_2^{-\frac{\epsilon_2^2}{2}}u^{-\epsilon_1\epsilon_2}\\
&\times\Big\{\Big(1 - k_1{}^{\frac{1}{2}}\big(d - 2 + k_1^{\epsilon_1}u^{\epsilon_2} + k_1^{-\epsilon_1}u^{-\epsilon_2} + N_{\text{s}} - 2\big)\Big)\\
&\quad\times\Big(1 - k_2{}^{\frac{1}{2}}\big(d - 2 + k_2^{\epsilon_2}u^{\epsilon_1} + k_2^{-\epsilon_2}u^{-\epsilon_1} + N_{\text{s}} - 2\big)\Big)\\
&\quad\times\Big(1 - \frac{y}{u}\big(k_1{}^{\frac{1}{2}}(\epsilon_2^2 - 2\alpha' m_2^2) + k_2{}^{\frac{1}{2}}(\epsilon_1^2 - 2\alpha' m_1^2)\big)\theta\phi\Big)\\
&\quad - k_1{}^{\frac{1}{2}}k_2{}^{\frac{1}{2}}\,y\,\big(d - 2 + k_1{}^{\epsilon_1}k_2{}^{\epsilon_2}u^{\epsilon_1+\epsilon_2} + k_1^{-\epsilon_1}k_2^{-\epsilon_2}u^{-\epsilon_1-\epsilon_2} + N_{\text{s}}\big)\\
&\quad + k_1{}^{\frac{1}{2}}k_2{}^{\frac{1}{2}}\,\frac{y}{u}\Big(d - 2 + N_{\text{s}} - 2\\
&\qquad + 2\big(\epsilon_1(k_1^{\epsilon_1}u^{\epsilon_2} - k_1^{-\epsilon_1}u^{-\epsilon_2}) + \epsilon_2(k_2^{\epsilon_2}u^{\epsilon_1} - k_2^{-\epsilon_2}u^{-\epsilon_1})\big)\\
&\qquad + k_1^{\epsilon_1}k_2^{-\epsilon_2}u^{\epsilon_1-\epsilon_2} + k_1^{-\epsilon_1}k_2^{-\epsilon_2}u^{\epsilon_2-\epsilon_1}\Big)\Big\} + \mathcal{O}(k_\mu)\,.
\end{aligned}
\tag{4.37}
$$

4.4 A symmetric parametrization

The integration measure in Eq. (4.37) is not, however, written in the most symmetric way possible, since two of the bosonic moduli $k_1{}^{\frac{1}{2}}$ and $k_2{}^{\frac{1}{2}}$ are multipliers of super-Schottky group generators, while the other modulus $u \equiv 1-y+\theta\phi$ is a cross-ratio of the fixed points. This makes it hard to find the field theory limit. To present the integration measure in a sufficiently symmetric way, we should try to write it so it is has the same form under permutations of the super-Schottky group elements $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_1^{-1}\mathbf{S}_2$. The reason for this is that the homology cycles a_1, a_2 and $(a_1^{-1}\cdot a_2)$ lift to these super-Schottky group elements on $\mathbf{CP}^{1|1} - \Lambda$, but any two of a_1, a_2 and $(a_1^{-1}\cdot a_2)$ (along with the appropriate

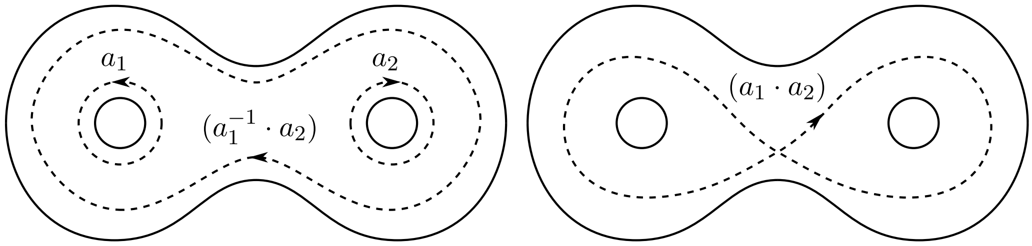


Figure 4.3: Two types of homology cycles on the double annulus.

choice of b-cycles) constitute a good canonical homology basis (see Fig. 4.3a). Any other homology cycle built out of a cycles will intersect itself (e.g. $(a_1 \cdot a_2)$, see Fig. 4.3b). Our choice of $\mathbf{S}_1$ and $\mathbf{S}_2$ as the generators is arbitrary, so in order to be modular invariant, the measure should be symmetric under permuting among $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_1^{-1}\mathbf{S}_2$.

A natural way to symmetrize the measure is therefore to use the multiplier of $\mathbf{S}_1^{-1}\mathbf{S}_2$ as the third bosonic modulus, instead of u. If we use ${k_3}^{\frac{1}{2}}$ to denote the smallest (in absolute value) eigenvalue of $\mathbf{S}_1\mathbf{S}_2^{-1}$ (so k_3 is its multiplier), then it can be found implicitly from the relation

$$y = \frac{(1-k_1)(1-k_2) + \theta\,\phi\left[(1-{k_1}^{\frac{1}{2}})(1-{k_2}^{\frac{1}{2}})(1+{k_1}^{\frac{1}{2}}{k_2}^{\frac{1}{2}})\right]}{1+k_1k_2-{k_1}^{\frac{1}{2}}{k_2}^{\frac{1}{2}}({k_3}^{\frac{1}{2}}+{k_3}^{-\frac{1}{2}})}\,. \tag{4.38}$$

Although these moduli are symmetric, they are not the most appropriate for investigating the symmetric degeneration, because the worldsheet pinches at two points on each of the cycles a_1, a_2 $(a_1^{-1} \cdot a_2)$. Instead, we should define the moduli p_i—or rather, their square roots, $\sqrt{p_i}$—with

$${k_1}^{\frac{1}{2}} = -\mathrm{e}^{\mathrm{i}\pi\varsigma_1}\sqrt{p_1}\sqrt{p_3} \qquad {k_2}^{\frac{1}{2}} = -\mathrm{e}^{\mathrm{i}\pi\varsigma_2}\sqrt{p_2}\sqrt{p_3} \qquad {k_3}^{\frac{1}{2}} = -\mathrm{e}^{\mathrm{i}\pi\varsigma_3}\sqrt{p_1}\sqrt{p_2}\,. \tag{4.39}$$

The $\sqrt{p_i}$'s are defined always to be positive, and the fact that the ${k_\alpha}^{\frac{1}{2}}$'s may be either positive or negative is allowed for by the inclusion of $\varsigma_i \in \mathbf{Z}_2$, the spin structures associated to the two b_i-cycles. ς_3 is the spin structure around the b_3 homology cycle and it is given simply by $\sigma_3 = \sigma_1 + \sigma_2 \pmod 2$.

In this way, each of the p_i behaves as a sewing parameter for one of the three NS degenerations.

To see that the $-$ signs in Eq. (4.39) are necessary, we note that whether ${k_1}^{\frac{1}{2}}$ and ${k_2}^{\frac{1}{2}}$ are both positive or are both negative, ${k_3}^{\frac{1}{2}}$ is negative, which is why there must be a $-$ sign in the third expression. Then by symmetry, there should be a $-$ sign in all three expressions.

The measure expressed in terms of the p_i is symmetric overall, but it's not symmetric within each sector ($\mathbf{F}_{\mathrm{gh}}$, $\mathbf{F}_{\mathrm{gl}}$, $\mathbf{F}_{\mathrm{scal}}$) under swapping $p_3 \leftrightarrow p_1$ or $p_3 \leftrightarrow p_2$, which it must be in order to obtain matching with QFT at the level of Feynman graphs. For example, with the moduli $(p_i|\theta,\phi)$, the factor within the square brackets of Eq. (4.4) is given as

$$\frac{\mathrm{d}k_1}{k_1^{3/2}}\frac{\mathrm{d}k_2}{k_2^{3/2}}\mathrm{d}\log y\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{\mathrm{d}p_1}{p_1^{3/2}}\frac{\mathrm{d}p_2}{p_2^{3/2}}\frac{\mathrm{d}p_3}{p_3}\,\mathrm{d}\theta\,\mathrm{d}\phi\,\frac{1-p_1p_2}{(1+p_3)(1+p_1p_2p_3)}\,. \tag{4.40}$$

The reason for this is that we are allowed to rescale the odd moduli by arbitrary functions, because the Berezinian of the transformation will cancel the rescaling and preserve the Berezin integral:

$$\mathrm{d}\tilde{\theta}\,\mathrm{d}\tilde{\phi}\,\tilde{\theta}\tilde{\phi} = \frac{\mathrm{d}\theta}{f_\theta(p_i|\theta,\phi)}\frac{\mathrm{d}\phi}{f_\phi(p_i|\theta,\phi)}\,\theta f_\theta(p_i|\theta,\phi)\phi f_\phi(p_i|\theta,\phi) = \mathrm{d}\theta\,\mathrm{d}\phi\,\theta\phi\,. \tag{4.41}$$

Even if there are natural odd moduli which make the symmetry manifest, a different choice of moduli which is related to the natural choice by an asymmetric rescaling will typically shuffle contributions between the various factors of the integrand, despite not changing the Berezin integral. Our odd moduli θ,ϕ must not be the natural choice, which is unsurprising because they are super-projective invariants of the fixed points of $\mathbf{S}_1$ and $\mathbf{S}_2$ only.

It is therefore necessary to find a pair of odd moduli which is invariant under permuting among $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_1^{-1}\mathbf{S}_2$. To do this, we can define

$$\hat{\theta}_{ij} \;=\; c_{ij}\,\Theta_{\mathbf{v}_i\mathbf{u}_i\mathbf{u}_j}\,, \qquad \hat{\phi}_{ij} \;=\; c_{ij}\,\Theta_{\mathbf{v}_i\mathbf{u}_i\mathbf{v}_j}\,, \tag{4.42}$$

for $(ij) = (12), (23), (31)$, where

$$c_{12} \;=\; \left[\left(1+\mathrm{e}^{\mathrm{i}\pi\varsigma_3}\sqrt{p_1}\sqrt{p_2}\right)\left(1-\mathrm{e}^{\mathrm{i}\pi\varsigma_1}\sqrt{p_1}\sqrt{p_3}\right)\left(1-\mathrm{e}^{\mathrm{i}\pi\varsigma_2}\sqrt{p_2}\sqrt{p_3}\right)\right]^{-1/2}\,, \tag{4.43}$$

with c_{23} and c_{31} obtained by permuting the indices (123). We've written $\varsigma_3 = \varsigma_1+\varsigma_2$, $\mathbf{u}_3$ and $\mathbf{v}_3$ label respectively the spin structure and the fixed points of the transformation $\mathbf{S}_1^{-1}\mathbf{S}_2$. In terms of these new Grassmann variables, Eq. (4.41), multiplied with the massless contribution of the ghost sectors in Eq. (4.8), reads

$$\prod_{i=1}^{3}\left[\frac{dp_i}{p_i^{3/2}}\,\frac{1+k_i^{\frac{1}{2}}}{\sqrt{1+p_i}}\right]d\hat{\theta}_{12}\,d\hat{\phi}_{12}\,\frac{1}{\sqrt{1+p_1p_2p_3}}\,. \tag{4.44}$$

One can check that

$$\mathrm{d}\hat{\theta}_{12}\,\mathrm{d}\hat{\phi}_{12} \;=\; \mathrm{d}\hat{\theta}_{23}\,\mathrm{d}\hat{\phi}_{23} \;=\; \mathrm{d}\hat{\theta}_{31}\,\mathrm{d}\hat{\phi}_{31}\,, \tag{4.45}$$

so that Eq. (4.44) is fully symmetric under permutations of the super-Schottky transformations $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_1^{-1}\mathbf{S}_2$ as expected. The various factors of the integrand can be expressed in terms of the variables $(p_i|\hat{\theta}_{12},\hat{\phi}_{12})$; to write them in this form, $\theta\phi$ can be expressed as follows:

$$\theta\phi = \sqrt{\frac{p_3(1+p_1)(1+p_2)}{(1+p_3)(1+p_1p_2p_3)}}(1+\mathrm{e}^{\mathrm{i}\pi\varsigma_3}\sqrt{p_1p_2})(1-\mathrm{e}^{\mathrm{i}\pi\varsigma_1}\sqrt{p_1p_3})(1-\mathrm{e}^{\mathrm{i}\pi\varsigma_2}\sqrt{p_2p_3})\,\hat{\theta}_{12}\hat{\phi}_{12} \tag{4.46}$$

$$= \sqrt{p_3}(1 + \mathrm{e}^{\mathrm{i}\pi\varsigma_3}\sqrt{p_1 p_2})\,\hat{\theta}_{12}\hat{\phi}_{12} + \mathcal{O}(p_i) \tag{4.47}$$

Up to $\mathcal{O}(p_i)$, the only contribution to $\mathbf{F}_{\mathrm{gh}}$ in Eq. (4.29) came from the zero modes; these have been included in Eq. (4.44), which becomes

$$\begin{aligned}\mathrm{d}\mu\,\mathbf{F}_{\mathrm{gh}}(\mathbf{m}) &= \prod_{i=1}^{3}\left[\frac{\mathrm{d}p_i}{p_i^{3/2}}\,\frac{1+k_i{}^{\frac{1}{2}}}{\sqrt{1+p_i}}\right]\mathrm{d}\hat{\theta}_{12}\,\mathrm{d}\hat{\phi}_{12}\,\frac{1}{\sqrt{1+p_1p_2p_3}} \\ &= \prod_{i=1}^{3}\left[\frac{\mathrm{d}p_i}{p_i^{3/2}}\right]\mathrm{d}\hat{\theta}_{12}\,\mathrm{d}\hat{\phi}_{12}\,(1-\mathrm{e}^{\mathrm{i}\pi\varsigma_1}\sqrt{p_1p_3}-\mathrm{e}^{\mathrm{i}\pi\varsigma_2}\sqrt{p_2p_3}-\mathrm{e}^{\mathrm{i}\pi\varsigma_3}\sqrt{p_1p_2})+\mathcal{O}(p_i)\,.\end{aligned} \tag{4.48}$$

We can make expressions more symmetric if we implicitly include the spin structure in the square roots, to be reinstated later when we carry out the GSO projection, by writing

$$\widetilde{\sqrt{p_1}} \equiv \mathrm{e}^{\mathrm{i}\pi\varsigma_1}\sqrt{p_1} \qquad \widetilde{\sqrt{p_2}} \equiv \mathrm{e}^{\mathrm{i}\pi\varsigma_2}\sqrt{p_2} \qquad \widetilde{\sqrt{p_3}} \equiv \sqrt{p_3} \qquad \widetilde{\sqrt{xy}} \equiv \widetilde{\sqrt{x}}\widetilde{\sqrt{y}} \tag{4.49}$$

so the right-hand side of Eq. (4.48) becomes

$$\prod_{i=1}^{3}\left[\frac{\mathrm{d}p_i}{p_i^{3/2}}\right]\mathrm{d}\hat{\theta}_{12}\,\mathrm{d}\hat{\phi}_{12}\left(1-\widetilde{\sqrt{p_1p_3}}-\widetilde{\sqrt{p_2p_3}}-\widetilde{\sqrt{p_1p_2}}\right)+\mathcal{O}(p_i)\,. \tag{4.50}$$

The twisted gluon sector comes from Eq. (4.32). To write it down in terms of the p_i, we need to make use of the substitution

$$u = p_3\big(1+\hat{\theta}_{12}\hat{\phi}_{12}(\widetilde{\sqrt{p_3}}-\widetilde{\sqrt{p_1}}-\widetilde{\sqrt{p_2}}+\widetilde{\sqrt{p_1p_2p_3}})\big)+\mathcal{O}(p_1,p_2)+\mathcal{O}(p_3^2), \tag{4.51}$$

which can be found by inserting Eq. (4.38) in $u \equiv 1-y+\theta\phi$ with the substitutions in Eq. (4.39) and using Eq. (4.42) to rewrite θ and ϕ in terms of $\hat{\theta}_{12}$ and $\hat{\phi}_{12}$. We find that the contribution from the orbital modes, coming from Eq. (4.34), is given by

$$\begin{aligned}1+\Big\{&\widetilde{\sqrt{p_1p_2}}\big[(p_1^{\epsilon_1}p_2^{-\epsilon_2}-p_1^{-\epsilon_1}p_2^{\epsilon_2})-\hat{\theta}_{12}\hat{\phi}_{12}\widetilde{\sqrt{p_3}}(\epsilon_1-\epsilon_2)(p_1^{\epsilon_1}p_2^{-\epsilon_2}-p_1^{-\epsilon_2}p_2^{\epsilon_2})\big] \\ &+\text{cyclic permutations of }(p_i,\epsilon_i)\Big\}+\mathcal{O}(p_i)\,.\end{aligned} \tag{4.52}$$

where $\epsilon_3 \equiv -\epsilon_1-\epsilon_2$. The exponential factor, given in Eq. (4.33), becomes

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}\pi\vec{\epsilon}\cdot\boldsymbol{\tau}\cdot\vec{\epsilon}} = p_1^{-\frac{\epsilon_1^2}{2}}p_2^{-\frac{\epsilon_2^2}{2}}p_3^{-\frac{\epsilon_3^2}{2}}\bigg(1-\frac{1}{2}\hat{\theta}_{12}\hat{\phi}_{12}\Big(&\widetilde{\sqrt{p_1}}(\epsilon_1^2-\epsilon_2^2-\epsilon_3^2)+\widetilde{\sqrt{p_1p_2p_3}}\,\epsilon_1^2 \\ &+\text{cyclic permutations of }(p_i,\epsilon_i)\Big)\bigg)+\mathcal{O}(p_i)\,.\end{aligned} \tag{4.53}$$

The twisted determinant $\det\left(\mathrm{Im}\,\boldsymbol{\tau}_{\vec{\epsilon}}\right)$, given in Eq. (3.59) can be expressed in these moduli by making the substitution Eq. (4.51) in the hypergeometric functions, which yields

$${}_2F_1(a-\epsilon_1,b-\epsilon_2,c-\epsilon_1-\epsilon_2,u) = 1+\frac{(a-\epsilon_1)(b-\epsilon_2)}{c-\epsilon_1-\epsilon_2}\,u+\mathcal{O}(u^2) \tag{4.54}$$

$$= 1+\mathcal{O}(p_i)\,, \tag{4.55}$$

and we find that $\det\left(\operatorname{Im}\boldsymbol{\tau}_{\vec{\epsilon}}\right)$ is equal to

$$
\begin{aligned}
\det\left(\operatorname{Im}\boldsymbol{\tau}_{\vec{\epsilon}}\right) = \frac{1}{4\pi^2}\Gamma(-\epsilon_1)\Gamma(-\epsilon_2)\Gamma(-\epsilon_3)\Bigg(& p_1^{\frac{\epsilon_1}{2}} p_2^{\frac{\epsilon_2}{2}} p_3^{\frac{\epsilon_3}{2}} \left(\epsilon_1 p_1^{-\frac{\epsilon_1}{2}} + \epsilon_2 p_2^{-\frac{\epsilon_2}{2}} + \epsilon_3 p_3^{-\frac{\epsilon_3}{2}}\right) \\
& + \hat{\theta_{12}}\hat{\phi}_{12}\Big\{\widetilde{\sqrt{p_1}}\Big(p_1^{\frac{\epsilon_1}{2}} p_2^{-\frac{\epsilon_2}{2}} p_3^{-\frac{\epsilon_3}{2}} + p_1^{-\frac{\epsilon_1}{2}} p_2^{\frac{\epsilon_2}{2}} p_3^{\frac{\epsilon_3}{2}}\Big) p_1^{\frac{\epsilon_1}{2}}\epsilon_2\epsilon_3 \\
& \qquad\qquad + \text{cyclic permutations of } (p_i, \epsilon_i)\Big\} \\
& + \hat{\theta_{12}}\hat{\phi}_{12}\,\widetilde{\sqrt{p_1 p_2 p_3}}\, p_1^{-3\frac{\epsilon_1}{2}} p_2^{-3\frac{\epsilon_2}{2}} p_3^{-3\frac{\epsilon_3}{2}} \Big\{2 p_2^{2\epsilon_2} p_3^{2\epsilon_3}\epsilon_1 (p_2^{\epsilon_2}\epsilon_2 + p_3^{\epsilon_3}\epsilon_3) \\
& \quad + p_1^{\epsilon_1}\left(\left(\epsilon_2 p_2^{\epsilon_2} + \epsilon_3 p_3^{\epsilon_3}\right)\left(\left(\epsilon_2+\epsilon_3\right) p_3^{\epsilon_3} p_2^{\epsilon_2} + 2\epsilon_3 p_2^{2\epsilon_2} + 2\epsilon_2 p_3^{2\epsilon_3}\right) p_1^{\epsilon_1} - \right. \\
& \quad \left(\epsilon_2\epsilon_3 p_3^{\epsilon_3} p_2^{\epsilon_2} + 2\epsilon_3\left(\epsilon_2+\epsilon_3\right) p_2^{2\epsilon_2} + 2\epsilon_2\left(\epsilon_2+\epsilon_3\right) p_3^{2\epsilon_3}\right) p_1^{2\epsilon_1} - \\
& \quad \left. p_2^{\epsilon_2} p_3^{\epsilon_3}\left(\left(2\epsilon_2^2 + 3\epsilon_3\epsilon_2 + 2\epsilon_3^2\right)\left(-p_3^{\epsilon_3}\right) p_2^{\epsilon_2} + \epsilon_1\epsilon_3 p_2^{2\epsilon_2} + \epsilon_1\epsilon_2 p_3^{2\epsilon_3}\right)\right)\Big\}\Bigg) \\
& + \left(\epsilon_i \leftrightarrow -\epsilon_\mu\right) + \mathcal{O}(p_i)\,.
\end{aligned} \tag{4.56}
$$

Next we want to find the ingredients for the untwisted gluon sector, which comes from Eq. (4.31), yielding

$$
\mathbf{F}_{\perp} = \left[\det(\operatorname{Im}\boldsymbol{\tau})\right]^{-\frac{d-2}{2}}\Big(1-(d-2)\big(\widetilde{\sqrt{p_1 p_3}} + \widetilde{\sqrt{p_2 p_3}} + \widetilde{\sqrt{p_1 p_2}}\big)\Big) + \mathcal{O}(p_i)\,, \tag{4.57}
$$

where the determinant of the period matrix, coming from Eq. (2.243), becomes

$$
\begin{aligned}
4\pi^2\det(\operatorname{Im}\boldsymbol{\tau}) = {} & \log p_1 \log p_2 + \log p_2 \log p_3 + \log p_3 \log p_1 \\
& - 2\,\hat{\theta}_{12}\hat{\phi}_{12}\Big\{(\widetilde{\sqrt{p_1}} - \widetilde{\sqrt{p_1 p_2 p_3}})\log p_1 \\
& \qquad + \text{ cyclic permutations of } (p_i, \epsilon_i)\Big\} + \mathcal{O}(p_i)\,.
\end{aligned} \tag{4.58}
$$

Finally, we need the ingredients for the scalar sector. The factor containing the contribution from the VEVs comes from Eq. (4.35) and is given by

$$
\begin{aligned}
\prod_{I=1}^{N_{\rm s}} \mathrm{e}^{2\pi\mathrm{i}\alpha'\vec{m}_I\cdot\boldsymbol{\tau}\cdot\vec{m}_I} = p_1^{\alpha' m_1^2} p_2^{\alpha' m_2^2} p_3^{\alpha' m_3^2}\Big(1 & + \alpha'\,\hat{\theta}_{12}\hat{\phi}_{12}\Big\{\widetilde{\sqrt{p_1}}(m_1^2 - m_2^2 - m_3^2) + \widetilde{\sqrt{p_1 p_2 p_3}}\, m_1^2 \\
& + \text{ cyclic permutations of } (p_i, m_i^2)\Big\}\Big) + \mathcal{O}(p_i)\,.
\end{aligned} \tag{4.59}
$$

The other factor in Eq. (4.36) is given as

$$
\prod_{\alpha}{}' \prod_{n=1}^{\infty}\Big(\frac{1-\mathrm{e}^{\mathrm{i}\pi\vec{\varsigma}\cdot\vec{N}_\alpha}\, k_\alpha^{n-\frac{1}{2}}}{1-k_\alpha^n}\Big)^{N_{\rm s}} = 1 + N_{\rm s}\Big(\widetilde{\sqrt{p_1 p_3}} + \widetilde{\sqrt{p_2 p_3}} + \widetilde{\sqrt{p_1 p_1}}\Big) + \mathcal{O}(p_i)\,. \tag{4.60}
$$

Combining Eq. (4.59) and Eq. (4.60), we get the total contribution from the scalar sector

$$
\begin{aligned}
\mathbf{F}_{\rm scal} = p_1^{\alpha' m_1^2} p_2^{\alpha' m_2^2} p_3^{\alpha' m_3^2}\Bigg(1 & + N_{\rm s}\Big(\widetilde{\sqrt{p_1 p_3}} + \widetilde{\sqrt{p_2 p_3}} + \widetilde{\sqrt{p_1 p_1}}\Big) \\
& + \alpha'\hat{\theta}_{12}\hat{\phi}_{12}\left(\big(\widetilde{\sqrt{p_1}}(m_1^2 - m_2^2 - m_3^2) + \text{ cyclic permutations of } (p_i, m_i^2)\big)\right.
\end{aligned}
$$

$$- (N_{\mathrm{s}} - 1)(m_1^2 + m_2^2 + m_3^2) \, \widetilde{\sqrt{p_1 p_2 p_3}} \Big) \bigg) + \mathcal{O}(p_i) \, . \tag{4.61}$$

Now we have expressed all of the factors of the integrand in terms of the symmetric p_i, $\hat{\theta}_{12}$, $\hat{\phi}_{12}$ moduli, so we can procede to compute the $\alpha' \to 0$ QFT limit of the amplitude, where we will find a 1–1 match between terms in the string integrand and Feynman diagrams.

4.4.1 Rewriting with field theory variables

To express the field theory limit of the amplitude in a form we can compare to our Feynman diagram calculations, we replace integration over the moduli p_i with integration over the dimensionful Schwinger parameters t_i via

$$p_i = \mathrm{e}^{-\frac{t_i}{\alpha'}} \, , \tag{4.62}$$

and re-express the twist ϵ_μ in terms of the background field strengths B_i via

$$\epsilon_i = 2\alpha' g B_i + \mathcal{O}({\alpha'}^3) \, , \qquad\qquad i = 1, 2, 3 \, . \tag{4.63}$$

The factor corresponding to the gluon modes parallel to the magnetic field can be found by multiplying Eq. (4.52) with Eq. (4.53) and dividing by Eq. (4.56), which yields

$$\begin{aligned} \mathbf{F}_{\parallel}^{(\vec{\epsilon})}(\mathbf{m}) = \frac{(2\pi\alpha')^2}{\Delta_F} \bigg(1 + \Big\{ & 2\widetilde{\sqrt{p_1 p_2}} \cosh(2gB_1 t_1 - 2gB_2 t_2) \\ & - \alpha' \, \hat{\theta}_{12} \hat{\phi}_{12} \, \frac{2}{\Delta_F} (\widetilde{\sqrt{p_1}} + \widetilde{\sqrt{p_1 p_2 p_3}}) \, \frac{\sinh(gB_1 t_1)}{gB_1} \cosh(2gB_1 t_1 - gB_2 t_2 - gB_3 t_3) \\ & + \text{cyclic permutations of } (p_i, t_i, B_i) \Big\} \bigg) + \mathcal{O}({\alpha'}^4) + \mathcal{O}(p_i) \, , \end{aligned} \tag{4.64}$$

where

$$\Delta_F = \frac{\cosh(gB_1 t_1 - gB_2 t_2 - gB_3 t_3)}{2g^2 B_2 B_3} + \text{cyclic permutations of } (t_i, B_i) \, , \tag{4.65}$$

while the contribution from the gluon modes perpendicular to the magnetic fields $\mathbf{F}_\perp$ becomes

$$\begin{aligned} \mathbf{F}_\perp = \frac{(2\pi\alpha')^{\frac{d}{2}-1}}{\Delta_0^{d/2-1}} \bigg(1 + (d-2) \Big\{ & \widetilde{\sqrt{p_1 p_3}} + \widetilde{\sqrt{p_2 p_3}} + \widetilde{\sqrt{p_1 p_2}} \\ & - \alpha' \hat{\theta}_{12} \hat{\phi}_{12} \, \frac{1}{\Delta_0} \Big(\widetilde{\sqrt{p_1}} \, t_1 + \widetilde{\sqrt{p_2}} \, t_2 + \widetilde{\sqrt{p_3}} \, t_3 \\ & + (d-3) \widetilde{\sqrt{p_1 p_2 p_3}} \, (t_1 + t_2 + t_3) \Big) \Big\} \bigg) + \mathcal{O}({\alpha'}^{\frac{d}{2}}) + \mathcal{O}(p_i) \, , \end{aligned} \tag{4.66}$$

where

$$\Delta_0 = t_1 t_2 + t_2 t_3 + t_3 t_1 = \lim_{B_i \to 0} \Delta_F \, . \tag{4.67}$$

The contribution from the D-brane world volume scalars comes from Eq. (4.61) and is equal to

$$\begin{aligned}\mathbf{F}_{\text{scal}}^{(\vec{m}_I)} &= \mathrm{e}^{-t_1 m_1^2}\mathrm{e}^{-t_2 m_2^2}\mathrm{e}^{-t_3 m_3^2}\Big(1 + N_{\mathrm{s}}\Big(\widetilde{\sqrt{p_1 p_3}} + \widetilde{\sqrt{p_2 p_3}} + \widetilde{\sqrt{p_1 p_1}}\Big) \\ &\quad + \alpha' \hat{\theta}_{12}\hat{\phi}_{12}\left(\widetilde{\sqrt{p_1}}(m_1^2 - m_2^2 - m_3^2) - (N_{\mathrm{s}} - 1)\, m_1^2 \widetilde{\sqrt{p_1 p_2 p_3}}\right. \\ &\qquad \left. + \text{ cyclic permutations of } (p_i, m_i^2)\right)\Big) + \mathcal{O}(p_i)\,. \end{aligned} \tag{4.68}$$

The final factor, including the integration measure and the contribution from the ghosts, is unchanged from Eq. (4.50).

The full integrand is the product of $\mathbf{F}_{\|}$ from Eq. (4.64), $\mathbf{F}_{\perp}$ from Eq. (4.66), $\mathbf{F}_{\text{scal}}$ from Eq. (4.68) and $\mathrm{d}\mu\, \mathbf{F}_{\text{gh}}$ from Eq. (4.50). We want to order the integrand by powers of $\sqrt{p_i}$ because negative powers will correspond to tachyons propagating in the ith leg, which we want to eliminate via the GSO projection, and positive powers correspond to massive states which we expect to decouple in the limit $\alpha' \to 0$. Let us define the power series coefficients $\widehat{\mathbf{F}}_{\|}^{ijk}$ via

$$\mathbf{F}_{\|} = \frac{(2\pi\alpha')^2}{\Delta_F} \sum_{i,j,k=0}^{\infty} \big(\widetilde{\sqrt{p_1}}\big)^i \big(\widetilde{\sqrt{p_2}}\big)^j \big(\widetilde{\sqrt{p_3}}\big)^k\, \widehat{\mathbf{F}}_{\|}^{ijk} \tag{4.69}$$

and similarly

$$\mathbf{F}_{\perp} = \frac{(2\pi\alpha')^{d-2}}{\Delta_0^{\frac{d-2}{2}}} \sum_{i,j,k=0}^{\infty} \big(\widetilde{\sqrt{p_1}}\big)^i \big(\widetilde{\sqrt{p_2}}\big)^j \big(\widetilde{\sqrt{p_3}}\big)^k\, \widehat{\mathbf{F}}_{\perp}^{ijk} \tag{4.70}$$

$$\mathbf{F}_{\text{scal}} = \prod_{i=1}^{3}\left[\mathrm{e}^{-t_i m_i^2}\right] \sum_{i,j,k=0}^{\infty} \big(\widetilde{\sqrt{p_1}}\big)^i \big(\widetilde{\sqrt{p_2}}\big)^j \big(\widetilde{\sqrt{p_3}}\big)^k\, \widehat{\mathbf{F}}_{\text{scal}}^{ijk} \tag{4.71}$$

$$\mathrm{d}\mu\, \mathbf{F}_{\text{gh}} = \prod_{i=1}^{3}\left[\frac{\mathrm{d}p_i}{p_i^{3/2}}\right] \mathrm{d}\hat{\theta}_{12}\, \mathrm{d}\hat{\phi}_{12} \sum_{i,j,k=0}^{\infty} \big(\widetilde{\sqrt{p_1}}\big)^i \big(\widetilde{\sqrt{p_2}}\big)^j \big(\widetilde{\sqrt{p_3}}\big)^k\, \widehat{\mathbf{F}}_{\text{gh}}^{ijk}\,. \tag{4.72}$$

We have then

$$\begin{aligned}\mathrm{d}\mu\, \mathbf{F}_{\text{gh}}\, \mathbf{F}_{\|}\, \mathbf{F}_{\perp}\, \mathbf{F}_{\text{scal}} &= (2\pi\alpha')^d \prod_{i=1}^{3}\left[\frac{\mathrm{d}p_i}{p_i^{3/2}}\, \mathrm{e}^{-t_i m_i^2}\right] \mathrm{d}\hat{\theta}_{12}\, \mathrm{d}\hat{\phi}_{12}\, \Delta_0^{-\frac{d-2}{2}}\, \Delta_F^{-1} \\ &\quad \times \sum_{i,j,k=0}^{\infty} \big(\widetilde{\sqrt{p_1}}\big)^i \big(\widetilde{\sqrt{p_2}}\big)^j \big(\widetilde{\sqrt{p_3}}\big)^k\, \widehat{\mathbf{F}}^{ijk} \end{aligned} \tag{4.73}$$

where

$$\widehat{\mathbf{F}}^{ijk} = \sum_{i_\ell, j_\ell, k_\ell = 0}^{\infty} \widehat{\mathbf{F}}_{\text{gh}}^{i_1 j_1 k_1}\, \widehat{\mathbf{F}}_{\|}^{i_2 j_2 k_2}\, \widehat{\mathbf{F}}_{\perp}^{i_3 j_3 k_3}\, \widehat{\mathbf{F}}_{\text{scal}}^{i_4 j_4 k_4}\, \delta_{(i_1+\ldots+i_4),i}\, \delta_{(j_1+\ldots+j_4),j}\, \delta_{(k_1+\ldots+k_4),k}\,. \tag{4.74}$$

4.5 The GSO projection

Now, if we view the amplitude as a sum over states propagating through the 3 pinched cycles, which map onto Feynman diagram edges, then the power of $\sqrt{p_i}$ corresponds to the mass level of the state in the ith edge. To see this, we note that we have

$$\frac{\mathrm{d}p_i}{p_i^{3/2}}\,(\sqrt{p_i})^n = -\frac{1}{\alpha'}\mathrm{d}t_i\,\mathrm{e}^{-\frac{n-1}{2\alpha'}t_i} \tag{4.75}$$

where $\mathrm{d}t_i\,\mathrm{e}^{-\frac{n-1}{2\alpha'}t_i}$ is a factor we would expect to see in a Schwinger-parameter propagator for a field with squared mass $m^2 = \frac{n-1}{2\alpha'}$. If $n = 0$ then the state propagating in the ith edge will have negative squared mass, i.e. it will be a tachyon and we will remove it via the GSO projection. If $n \geq 2$ then the state will have squared mass $m^2 = \frac{n-1}{2\alpha'} > 0$, which will become infinitely large as $\alpha' \to 0$ and therefore the state will decouple. For $n = 1$, we find $m^2 = 0$; once the tachyons are removed and we take the limit $\alpha' \to 0$, we expect these states to give the only non-vanishing contribution to the amplitude, and so the sole non-vanishing contribution to the field theory limit of Eq. (4.73) comes from $\widehat{\mathbf{F}}^{111}$.

A cursory look at $\mathbf{F}_{\parallel}$ in Eq. (4.64), $\mathbf{F}_{\perp}$ in Eq. (4.66), $\mathbf{F}_{\mathrm{scal}}$ in Eq. (4.68) and $\mathrm{d}\mu\,\mathbf{F}_{\mathrm{gh}}$ in Eq. (4.50), might suggest that tachyons can propagate simultaneously in an arbitrary number of edges because we can find terms proportional to $1, \widetilde{\sqrt{p_1}}, \widetilde{\sqrt{p_1 p_2}}, \widetilde{\sqrt{p_1 p_2 p_3}}, \ldots$ and so on, which correspond to 3 tachyon edges, 2 tachyons, 1 tachyon and 0 tachyons, respectively. A closer inspection shows, however, that the nilpotent object $\hat{\theta}_{12}\hat{\phi}_{12}$ multiplies a term if and only if there are an *odd* number of $\widetilde{\sqrt{p_i}}$'s (this property is preserved when we multiply terms together). Since the Berezin integral over $\mathrm{d}\hat{\theta}_{12}\,\mathrm{d}\hat{\phi}_{12}$ picks out the coefficient of $\hat{\theta}_{12}\hat{\phi}_{12}$, it follows that after carrying out the Berezin integral, each term must contain an odd number of $\widetilde{\sqrt{p_i}}$'s.

After carrying out the Berezin integral but before carrying out the GSO projection, the integrand truncated to $\mathcal{O}(\sqrt{p_i})$ will therefore be a sum of four terms whose coefficients are

$$\widetilde{\sqrt{p_1}} = \mathrm{e}^{\mathrm{i}\pi\varsigma_1}\sqrt{p_1}\,; \quad \widetilde{\sqrt{p_2}} = \mathrm{e}^{\mathrm{i}\pi\varsigma_2}\sqrt{p_2}\,; \quad \widetilde{\sqrt{p_3}} = \sqrt{p_3}\,; \quad \widetilde{\sqrt{p_1p_2p_3}} = \mathrm{e}^{\mathrm{i}\pi(\varsigma_1+\varsigma_2)}\sqrt{p_1p_2p_3}\,. \tag{4.76}$$

There are four possible spin structures along the a-cycles:

$$(\varsigma_1, \varsigma_2) \in \{(0,0), (1,0), (0,1), (1,1)\} = \mathbb{Z}_2^2\,. \tag{4.77}$$

We need to sum the amplitude over the four spin structures $\vec{\varsigma}$ with the appropriate signs $\sigma(\vec{\varsigma})$. The first three terms in Eq. (4.76) correspond to tachyons propagating in loops; since we wish to excise tachyons from the spectrum, the signs needs to be chosen so that these terms vanish. We have

$$\sum_{g\in\mathbb{Z}_2^2} \sigma(\vec{\varsigma})\,\mathrm{e}^{\mathrm{i}\pi\varsigma_1}\sqrt{p_1} = \sqrt{p_1}\big(\sigma(0,0) - \sigma(1,0) + \sigma(0,1) - \sigma(1,1)\big) \tag{4.78}$$

$$\sum_{g\in\mathbb{Z}_2^2} \sigma(\vec{\varsigma})\,\mathrm{e}^{\mathrm{i}\pi\varsigma_2}\sqrt{p_2} = +\sqrt{p_2}\big(\sigma(0,0) + \sigma(1,0) - \sigma(0,1) - \sigma(1,1)\big) \tag{4.79}$$

$$\sum_{g\in\mathbb{Z}_2^2} \sigma(\vec{\varsigma})\,\sqrt{p_3} = \sqrt{p_3}\big(\sigma(0,0)+\sigma(1,0)+\sigma(0,1)+\sigma(1,1)\big)\,, \tag{4.80}$$

and we need all of these to vanish, so we need to set

$$\sigma(0,0) = \sigma(1,1) = -\sigma(1,0) = -\sigma(0,1)\,. \tag{4.81}$$

A suitably normalized GSO projection is given, then, by the weighting $\sigma(\vec{\varsigma}) = \frac{1}{4}e^{i\pi(\varsigma_1+\varsigma_2)}$. For the remaining term, this gives

$$\sum_{g\in\mathbb{Z}_2^2} \sigma(\vec{\varsigma})\,\mathrm{e}^{\mathrm{i}\pi(\varsigma_1+\varsigma_2)}\sqrt{p_1p_2p_3} = \sqrt{p_1p_2p_3}\,, \tag{4.82}$$

and so after carrying out the Berezin integration and the GSO projection, we are left only with the massless sector, entering Eq. (4.73) via the term $\widehat{\mathbf{F}}^{111}$.

The amplitude is obtained by integrating the measure $\mathrm{d}\mu\,\mathbf{F}_{\mathrm{gh}}\,\mathbf{F}_{\|}\,\mathbf{F}_{\perp}\,\mathbf{F}_{\mathrm{scal}}$ from Eq. (4.73) and multiplying by an overall factor

$$A = C_2\prod_{i=1}^{2}\Big(\frac{1}{\cos(\pi\epsilon_i)}\Big)\int \mathrm{d}\mu\,\mathbf{F}_{\mathrm{gh}}\,\mathbf{F}_{\|}\,\mathbf{F}_{\perp}\,\mathbf{F}_{\mathrm{scal}} \tag{4.83}$$

where C_h, the normalization factor for an h-loop string amplitude in terms of the d-dimensional Yang-Mills coupling g, is calculated in Appendix A of [100] and is given for $h=2$ by

$$C_2 = \frac{1}{(2\pi)^{2d}}(2\alpha')^{-d}g^2(\alpha')^2 = \frac{g^2}{(4\pi)^d}\frac{(\alpha')^2}{(2\pi\alpha')^d}\,. \tag{4.84}$$

The other prefactor in Eq. (4.83) is equal to $\big(\cos(\pi\epsilon_1)\cos(\pi\epsilon_2)\big)^{-1} = 1+\mathcal{O}(\alpha'^2)$ so we can neglect it in the field theory limit. Plugging Eq. (4.73) into Eq. (4.83) with the GSO projection carried out, we find

$$A = \frac{g^2(\alpha')^2}{(4\pi)^d}\int\prod_{i=1}^{3}\left[\frac{\mathrm{d}p_i}{p_i}\,\mathrm{e}^{-t_i m_i^2}\right]\mathrm{d}\hat{\theta}_{12}\,\mathrm{d}\hat{\phi}_{12}\,\Delta_0^{-\frac{d-2}{2}}\,\Delta_F^{-1}\widehat{\mathbf{F}}^{111} + \mathcal{O}(\alpha') + \mathcal{O}(\mathrm{e}^{-1/\alpha'}) \tag{4.85}$$

Using $\mathrm{d}p_i/p_i = -\mathrm{d}t_i/\alpha'$, we find that the QFT limit is given by

$$A_{\mathrm{QFT}} = \lim_{\alpha'\to 0} A \tag{4.86}$$

$$= \frac{g^2}{(4\pi)^d}\int\frac{\prod_{i=1}^{3}\mathrm{d}t_i\,\mathrm{e}^{-t_i m_i^2}}{\Delta_0^{\frac{d-2}{2}}\,\Delta_F}\Big(-\frac{1}{\alpha'}\int\mathrm{d}\hat{\theta}_{12}\,\mathrm{d}\hat{\phi}_{12}\,\widehat{\mathbf{F}}^{111}\Big)\,, \tag{4.87}$$

If we write

$$\mathbf{f}_{\mathrm{gh}}^{i_1j_1k_1}\,\mathbf{f}_{\|}^{i_2j_2k_2}\,\mathbf{f}_{\perp}^{i_3j_3k_3}\,\mathbf{f}_{\mathrm{scal}}^{i_4j_4k_4} = -\frac{1}{\alpha'}\int\mathrm{d}\hat{\theta}_{12}\,\mathrm{d}\hat{\phi}_{12}\,\widehat{\mathbf{F}}_{\mathrm{gh}}^{i_1j_1k_1}\,\widehat{\mathbf{F}}_{\|}^{i_2j_2k_2}\,\widehat{\mathbf{F}}_{\perp}^{i_3j_3k_3}\,\widehat{\mathbf{F}}_{\mathrm{scal}}^{i_4j_4k_4}\,, \tag{4.88}$$

with

$$\mathbf{f}^{000}_{\rm gh} = \mathbf{f}^{000}_{\parallel} = \mathbf{f}^{000}_{\perp} = \mathbf{f}^{000}_{\rm scal} = 1\,, \tag{4.89}$$

then from Eq. (4.74) we see we have

$$\begin{aligned} -\frac{1}{\alpha}\int \mathrm{d}\hat{\theta}_{12}\,\mathrm{d}\hat{\phi}_{12}\,\widehat{\mathbf{F}}^{ijk} = \sum_{i_\ell,j_\ell,k_\ell=0}^{\infty} &\Big\{\mathbf{f}^{i_1j_1k_1}_{\rm gh}\,\mathbf{f}^{i_2j_2k_2}_{\parallel}\,\mathbf{f}^{i_3j_3k_3}_{\perp}\,\mathbf{f}^{i_4j_4k_4}_{\rm scal} \\ &\times\,\delta_{(i_1+\ldots+i_4),i}\,\delta_{(j_1+\ldots+j_4),j}\,\delta_{(k_1+\ldots+k_4),k}\Big\}\,. \end{aligned} \tag{4.90}$$

Substituting this for $\widehat{\mathbf{F}}^{111}$ in Eq. (4.87), we get

$$\begin{aligned} A_{\rm QFT} &= \sum_{i_\ell,j_\ell,k_\ell=0}^{1}\frac{g_d^2}{(4\pi)^d}\int\frac{\prod_{i=1}^3 \mathrm{d}t_i\,\mathrm{e}^{-t_i m_i^2}}{\Delta_0^{\frac{d-2}{2}}\,\Delta_F}\Big\{\mathbf{f}^{i_1j_1k_1}_{\rm gh}\,\mathbf{f}^{i_2j_2k_2}_{\parallel}\,\mathbf{f}^{i_3j_3k_3}_{\perp}\,\mathbf{f}^{i_4j_4k_4}_{\rm scal} \\ &\qquad\times\,\delta_{(i_1+\ldots+i_4),1}\,\delta_{(j_1+\ldots+j_4),1}\,\delta_{(k_1+\ldots+k_4),1}\Big\} \quad (4.91)\\ &= \frac{g_d^2}{(4\pi)^d}\int\frac{\prod_{i=1}^3 \mathrm{d}t_i\,\mathrm{e}^{-t_i m_i^2}}{\Delta_0^{\frac{d-2}{2}}\,\Delta_F}\Big\{\mathbf{f}^{111}_{\parallel} + \mathbf{f}^{110}_{\parallel}\mathbf{f}^{001}_{\perp} + \ldots(25\text{ more terms})\Big\}\,. \quad (4.92) \end{aligned}$$

We can read off the various terms in the integrand by picking the coefficients of the appropriate $\widetilde{\sqrt{p_i}}$'s from $\mathbf{F}_\parallel$ in Eq. (4.64), $\mathbf{F}_\perp$ in Eq. (4.66), $\mathbf{F}_{\rm scal}$ in Eq. (4.68) and $\mathrm{d}\mu\,\mathbf{F}_{\rm gh}$ in Eq. (4.50) and writing down the coefficient of $\hat{\theta}_{12}\,\hat{\phi}_{12}$, divided by α'. We have

$$\begin{aligned} \mathbf{f}^{111}_{\parallel} &= \frac{2}{\Delta_F}\frac{\sinh(gB_1t_1)}{gB_1}\cosh(2gB_1t_1 - gB_2t_2 - gB_3t_3) \quad (4.93)\\ &\qquad + \text{cyclic permutations of }(t_i, B_i)\,,\\ \mathbf{f}^{110}_{\parallel}\mathbf{f}^{001}_{\perp} &= \frac{2(d-2)}{\Delta_0}\cosh(2gB_1t_1 - 2gB_2t_2)\,t_3 \quad (4.94)\\ \mathbf{f}^{001}_{\parallel}\mathbf{f}^{110}_{\perp} &= \frac{2(d-2)}{\Delta_F}\frac{\sinh(gB_3t_3)}{gB_3}\cosh(2gB_3t_3 - gB_1t_1 - gB_2t_2) \quad (4.95)\\ \mathbf{f}^{111}_{\perp} &= \frac{(d-2)(d-3)}{\Delta_0}(t_1+t_2+t_3) \quad (4.96)\\ \mathbf{f}^{110}_{\rm gh}\mathbf{f}^{001}_{\parallel} &= -\frac{2}{\Delta_F}\frac{\sinh(gB_3t_3)}{gB_3}\cosh(2gB_3t_3 - gB_1t_1 - gB_2t_2)\,, \quad (4.97)\\ \mathbf{f}^{110}_{\rm gh}\,\mathbf{f}^{001}_{\perp} &= -\frac{d-2}{\Delta_0}t_3\,, \quad (4.98)\\ \mathbf{f}^{110}_{\rm gh}\mathbf{f}^{001}_{\rm scal} &= m_3^2 - m_1^2 - m_2^2\,, \quad (4.99)\\ \mathbf{f}^{110}_{\rm scal}\mathbf{f}^{001}_{\parallel} &= \frac{2N_{\rm s}}{\Delta_F}\frac{\sinh(gB_3t_3)}{gB_3}\cosh(2gB_3t_3 - gB_1t_1 - gB_2t_2)\,, \quad (4.100)\\ \mathbf{f}^{110}_{\rm scal}\mathbf{f}^{001}_{\perp} &= \frac{(d-2)N_{\rm s}}{\Delta_0}t_3\,, \quad (4.101)\\ \mathbf{f}^{110}_{\parallel}\mathbf{f}^{001}_{\rm scal} &= 2(m_1^2+m_2^2-m_3^2)\cosh(2gB_1t_1 - 2gB_2t_2)\,, \quad (4.102)\\ \mathbf{f}^{110}_{\perp}\mathbf{f}^{001}_{\rm scal} &= (d-2)(m_1^2+m_2^2-m_3^2)\,, \quad (4.103)\\ \mathbf{f}^{111}_{\rm scal} &= (N_{\rm s}-1)(m_1^2+m_2^2+m_3^2)\,. \quad (4.104) \end{aligned}$$

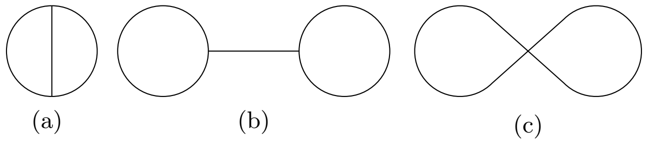


Figure 4.4: Possible topologies for 2-loop vacuum graphs with 3- and 4-point vertices.

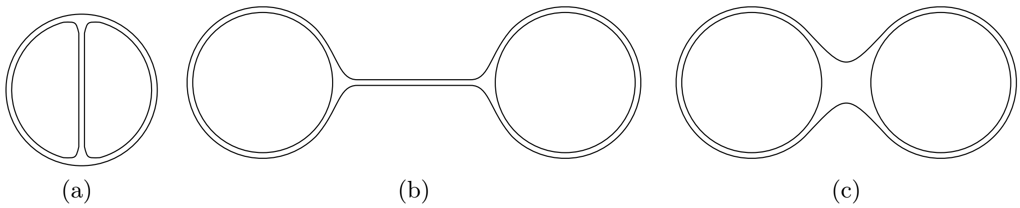


Figure 4.5: Three possible topologies for a double annulus with at least two pinched cycles: non-separating (Fig. 4.4a), separating (Fig. 4.5b) a,nd incomplete (Fig. 4.5c).

The other terms in the integrand can be obtained from these by cyclic symmetry, i.e. $\mathbf{f}_{\parallel}^{101}\mathbf{f}_{\perp}^{101}$ can be obtained from $\mathbf{f}_{\parallel}^{110}\mathbf{f}_{\perp}^{001}$ in Eq. (4.94) by cycling $(t_1, t_2, t_3; B_1, B_2, B_3) \to (t_3, t_1, t_2; B_3, B_1, B_2)$, and so on.

Eq. (4.91) doesn't give the full 2-loop vacuum amplitude with this topology, because it is calculated with the worldsheet boundaries on 3 fixed D-branes (*i.e.* fixed Chan-Paton factors). To calculate the full amplitude, it is necessary to sum over all possible choices of D-branes for each of the three boundaries. Since the D-branes are distinguished only by the strengths of the background fields on their worldvolumes B^i and their relative positions in the transverse directions Y_I^{ij}, this is equivalent to summing the preceding expressions over all possible values of B^{ij} and m_I^{ij}. That is to say, the full contribution to the vacuum amplitude from diagrams with this topology will be given by the sum

$$\mathcal{A}_{\mathrm{QFT}} = \sum_{i,j,k=1}^{N} \left(A_{\mathrm{QFT}} \Big|_{\substack{B_1=B^{ij},\ B_2=B^{jk},\ B_3=B^{ki},\\ m_1^2=m_{ij}^2,\ m_2^2=m_{jk}^2,\ m_3^2=m_{ki}^2}} \right), \tag{4.105}$$

where a string stretched between the ith and jth D-branes feels a background field proportional to $B^{ij} \equiv B^i - B^j$ and has a classical mass of $|m_{ij}|$ (recall Eq. (4.3)).

4.6 Incomplete and separating degenerations

The integral calculated in the previous section corresponds to the sum of all two-loop QFT diagrams with the topology of figs. Fig. 4.4a. The full 2-loop vacuum amplitude includes other Feynman diagrams with the topology of Fig. 4.4c, i.e. with a quartic vertex, and 1-particle-reducible (1PR) diagrams with the topology of Fig. 4.4b. We don't need to calculate the 1PR graphs to compute the effective action, but it is crucial to include the diagrams with quartic vertices. There are two topologically distinct ways a double-annulus worldsheet can totally degenerate: the non-separating degeneration of fig. 4.5a and the separating degeneration of fig. 4.5b; the QFT limit of the string amplitude also gains contributions from the incomplete degeneration in fig. 4.5c where only two homology cycles are pinched; this interpolates between the two total degenerations. The separating, non-separating and incomplete degenerations all come from the region of super-moduli space in which the two Schottky multipliers k_1 and k_2 are small; the degenerations differ in the value of the third bosonic modulus u (or y), which is a super-projective invariant of the four Schottky fixed points. The separating degeneration corresponds to the limit $y \to 0$ while the non-separating degeneration corresponds to the limit $u \to 0$, while the incomplete degeneration comes from the rest of the moduli space.

When we truncated the measure to low order in the p_is, we actually excised the contribution from the incomplete degeneration and from the separating degeneration. It is therefore necessary to retrace our steps to the measure expressed in terms of k_μ and u (or y) in Eq. (4.37). Not counting the factors on the first two lines, the expression contains 5 types of terms: some proportional to either $k_1^{\frac{1}{2}}$ or $k_2^{\frac{1}{2}}$ on their own; some proportional to $-\frac{y}{u}k_1^{\frac{1}{2}}k_2^{\frac{1}{2}} = k_3^{\frac{1}{2}} + \mathcal{O}(k_\mu)$; some proportional to $yk_1^{\frac{1}{2}}k_2^{\frac{1}{2}} = k(\mathbf{S}_1\mathbf{S}_2)^{\frac{1}{2}} + \mathcal{O}(k_\mu)$, and some proportional to $k_1^{\frac{1}{2}}k_2^{\frac{1}{2}}$ without these prefactors. Retracing the calculation of the previous section, we find that the terms proportional to $k_1^{\frac{1}{2}}$, $k_2^{\frac{1}{2}}$ and $-\frac{y}{u}k_1^{\frac{1}{2}}k_2^{\frac{1}{2}}$ are transformed into terms proportional to $\sqrt{p_1p_3}$, $\sqrt{p_2p_3}$ and $\sqrt{p_1p_2}$, respectively. The other two types of terms — those which originate from $\mathbf{S}_1\mathbf{S}_2$ or those which are a product of a factor coming from $\mathbf{S}_1$ and a factor coming from $\mathbf{S}_2$ — disappear as $\mathcal{O}(p_i)$ corrections.

They do, however, contribute to the QFT limit; we hope that they give the remaining Feynman diagrams. Since the worldsheets no longer have a three-fold symmetry, we don't have to worry about needing a three-fold symmetry parametrization. The idea, then, is to remove from Eq. (4.37) the terms proportional to $k_1^{\frac{1}{2}}$, $k_2^{\frac{1}{2}}$ or $k_3^{\frac{1}{2}}$ which we've already accounted for in the p_i parametrization, relate k_1 and k_2 to Schwinger parameters as we did with the p_i for the non-separating degeneration, and integrate over the third bosonic modulus. We split $[\mathrm{d}\mathbf{m}]_2$ from Eq. (4.37) up into a part $[\mathrm{d}\mathbf{m}]_2^{\text{non-sep}}$ whose QFT limit is already accounted for, and another part $[\mathrm{d}\mathbf{m}]_2^{\text{rem}}$ which vanishes at $\mathcal{O}(p_i)$.

$$[\mathrm{d}\mathbf{m}]_2^0 = [\mathrm{d}\mathbf{m}]_2^{\text{non-sep}} + [\mathrm{d}\mathbf{m}]_2^{\text{rem}} \tag{4.106}$$

$$\begin{aligned}[\mathrm{d}\mathbf{m}]_2^{\text{rem}} = {}& \frac{\mathrm{d}k_1}{k_1^{3/2}}\frac{\mathrm{d}k_2}{k_2^{3/2}}\frac{\mathrm{d}u}{y}\,\mathrm{d}\theta\,\mathrm{d}\phi\,\frac{\big[\det(\operatorname{Im}\boldsymbol{\tau})\big]^{-\frac{d-2}{2}}}{\det(\operatorname{Im}\boldsymbol{\tau}_{\vec{\epsilon}})}k_1^{\frac{1}{2}}k_2^{\frac{1}{2}} \\ & \times k_1^{\alpha' m_1^2}k_2^{\alpha' m_2^2}u^{\alpha'(m_3^2-m_1^2-m_2^2)}k_1^{-\frac{\epsilon_1^2}{2}}k_2^{-\frac{\epsilon_2^2}{2}}u^{-\epsilon_1\epsilon_2} \\ & \times\Big\{\big(d-2+k_1^{\epsilon_1}u^{\epsilon_2}+k_1^{-\epsilon_1}u^{-\epsilon_2}+N_{\mathrm{s}}-2\big) \\ & \qquad\times\big(d-2+k_2^{\epsilon_2}u^{\epsilon_1}+k_2^{-\epsilon_2}u^{-\epsilon_1}+N_{\mathrm{s}}-2\big) \\ & \quad - y\big(d-2+k_1^{\epsilon_1}k_2^{\epsilon_2}u^{\epsilon_1+\epsilon_2}+k_1^{-\epsilon_1}k_2^{-\epsilon_2}u^{-\epsilon_1-\epsilon_2}+N_{\mathrm{s}}\big)\Big\}+\mathcal{O}(k_\mu)\,.\end{aligned} \tag{4.107}$$

$$= \prod_{i=1}^{3}\left[\frac{\mathrm{d}p_i}{p_i^{3/2}}\right]\mathrm{d}\hat{\theta}_{12}\,\mathrm{d}\hat{\phi}_{12}\,\frac{\big[\det(\operatorname{Im}\boldsymbol{\tau})\big]^{-\frac{d-2}{2}}}{\det(\operatorname{Im}\boldsymbol{\tau}_{\vec{\epsilon}})}\Big(0+\mathcal{O}(p_i)\Big)\,. \tag{4.108}$$

To find the field theory limit of Eq. (4.107), we replace integration over the two multipliers

k_μ with integration over dimensionful Schwinger parameters similarly to Eq. (4.62) but here we have only two:

$$k_\mu = \mathrm{e}^{-\frac{t_\mu}{\alpha'}}\,, \qquad\qquad \mu = 1,\,2\,; \tag{4.109}$$

we also replace ϵ_μ according to Eq. (4.63). Making these substitutions in $\det(\mathrm{Im}\,\boldsymbol{\tau})$ as given in Eq. (2.243) and $\det(\mathrm{Im}\,\boldsymbol{\tau}_{\vec{\epsilon}})$ as given in Eq. (3.59), we find

$$\Big[\det\left(\mathrm{Im}\,\boldsymbol{\tau}\right)\Big]^{-\frac{d-2}{2}} = \Big(\frac{(2\pi\alpha')^2}{t_1 t_2}\Big)^{\frac{d}{2}-1} + \mathcal{O}\big((\alpha')^{d-1}\big) \tag{4.110}$$

$$\det\left(\mathrm{Im}\,\boldsymbol{\tau}_{\vec{\epsilon}}\right)^{-1} = (2\pi\alpha')^2 \frac{gB_1}{\sinh(gB_1 t_1)}\frac{gB_2}{\sinh(gB_2 t_2)} + \mathcal{O}\big((\alpha')^3\big)\,. \tag{4.111}$$

Performing the sum over spin structure as in Eq. (4.81) (where we recall that the spin structures are not exhibited explicitly but are implicit in the signs of $k_1{}^{\frac{1}{2}}$ and $k_2{}^{\frac{1}{2}}$), the integrals over the multipliers $\mathrm{d}k_\mu$ become

$$\sum_{\vec{\varsigma}} \frac{1}{4}\mathrm{e}^{\mathrm{i}\,\pi(\varsigma_1+\varsigma_2)}\,\frac{\mathrm{d}k_1}{k_1^{3/2}}\,\frac{\mathrm{d}k_2}{k_2^{3/2}}\,k_1^{\frac{1}{2}}k_2^{\frac{1}{2}} = \frac{1}{(\alpha')^2}\,\mathrm{d}t_1\,\mathrm{d}t_2\,. \tag{4.112}$$

The factor in the second line of Eq. (4.107) becomes

$$k_1^{\alpha' m_1^2} k_2^{\alpha' m_2^2} u^{\alpha'(m_3^2 - m_1^2 - m_2^2)} k_1^{-\frac{\epsilon_1^2}{2}} k_2^{-\frac{\epsilon_2^2}{2}} u^{-\epsilon_1\epsilon_2} = \mathrm{e}^{-t_1 m_1^2 - t_2 m_2^2} + \mathcal{O}(\alpha')\,, \tag{4.113}$$

and we have

$$k_1^{\epsilon_1} u^{\epsilon_2} + k_1^{-\epsilon_1} u^{-\epsilon_2} = 2\cosh(2gB_1t_1) + \mathcal{O}(\alpha') \tag{4.114}$$

$$k_2^{\epsilon_2} u^{\epsilon_1} + k_2^{-\epsilon_2} u^{-\epsilon_1} = 2\cosh(2gB_2t_2) + \mathcal{O}(\alpha') \tag{4.115}$$

$$k_1^{\epsilon_1} k_2^{\epsilon_2} u^{\epsilon_1+\epsilon_2} + k_1^{-\epsilon_1} k_2^{-\epsilon_2} u^{-\epsilon_1-\epsilon_2} = 2\cosh(2gB_1t_1 + 2gB_2t_2) + \mathcal{O}(\alpha')\,. \tag{4.116}$$

We can rewrite Eq. (4.107), then, as

$$\begin{aligned}[\mathrm{d}\mathbf{m}]_2^{\mathrm{rem}} = \frac{(2\pi\alpha')^d}{(\alpha')^2}&\Big[\prod_{i=1}^{2}\frac{\mathrm{d}t_i\,\mathrm{e}^{-t_i m_i^2}\,gB_i}{t_i^{d/2-1}\,\sinh(gB_i t_i)}\Big]\,\mathrm{d}u\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &\times\Big\{\frac{1}{y}\big(d-2+2\cosh(2gB_1t_1)+N_{\mathrm{s}}-2\big) \\ &\qquad\qquad \times\big(d-2+2\cosh(2gB_2t_2)+N_{\mathrm{s}}-2\big) \\ &\quad -\big(d-2+2\cosh(2gB_1t_1+2gB_2t_2)+N_{\mathrm{s}}\big)\Big\} + \mathcal{O}(\mathrm{e}^{-t_\mu/\alpha'}) + \mathcal{O}(\alpha')\,.\end{aligned} \tag{4.117}$$

Inserting this measure in Eq. (4.83) with the normalization constant given in Eq. (4.84) as we did with $[\mathrm{d}\mathbf{m}]_2^{\text{non-sep}}$, we obtain

$$A_{\mathrm{QFT}}^{\mathrm{rem}} = \lim_{\alpha'\to 0}\frac{g^2}{(4\pi)^d}\frac{(\alpha')^2}{(2\pi\alpha')^d}\,[\mathrm{d}\mathbf{m}]_2^{\mathrm{rem}} \tag{4.118}$$

$$
\begin{aligned}
&= \frac{g^2}{(4\pi)^d}\int_0^\infty \Big[\prod_{i=1}^{2}\frac{\mathrm{d}t_i\,\mathrm{e}^{-t_i m_i^2}\,gB_i}{t_i^{d/2-1}\,\sinh(gB_i t_i)}\Big] \\
&\quad\times\Big\{I_1\,\big(d-2+2\cosh(2gB_1t_1)+N_\mathrm{s}-2\big) \\
&\qquad\qquad\times\big(d-2+2\cosh(2gB_2t_2)+N_\mathrm{s}-2\big) \\
&\quad\ - I_2\big(d-2+2\cosh(2gB_1t_1+2gB_2t_2)+N_\mathrm{s}\big)\Big\}\,.
\end{aligned}
\tag{4.119}
$$

Where I_1 and I_2 denote the two Berezin integrals

$$
I_1 = \int_{\widehat{\mathfrak{M}}_{u|\theta\phi}} \frac{\mathrm{d}u}{y}\,\mathrm{d}\theta\,\mathrm{d}\phi \qquad\qquad I_2 = \int_{\widehat{\mathfrak{M}}_{u|\theta\phi}} \mathrm{d}u\,\mathrm{d}\theta\,\mathrm{d}\phi\,. \tag{4.120}
$$

These can be calculated with Stokes' theorem for a supermanifold with a boundary (see section 3.4 of [83]). We can reexpress them in terms of integral forms:

$$
I_1 = \int_{\widehat{\mathfrak{M}}_{u|\theta\phi}} \mathrm{d}u\,\frac{1}{y}\,\delta^2(\mathrm{d}\theta,\,\mathrm{d}\phi) \qquad\qquad I_2 = \int_{\widehat{\mathfrak{M}}_{u|\theta\phi}} \mathrm{d}u\,\delta^2(\mathrm{d}\theta,\,\mathrm{d}\phi)\,. \tag{4.121}
$$

but these integral forms can be expressed as exterior derivatives:

$$
\frac{\mathrm{d}u}{y}\,\delta^2(\mathrm{d}\theta,\,\mathrm{d}\phi) = \mathrm{d}\nu_1\,; \qquad\qquad \mathrm{d}u\,\delta^2(\mathrm{d}\theta,\,\mathrm{d}\phi) = \mathrm{d}\nu_2\,; \tag{4.122}
$$

$$
\nu_1 = -\log(y)\,\delta^2(\mathrm{d}\theta,\,\mathrm{d}\phi)\,, \qquad\qquad \nu_2 = u\,\delta^2(\mathrm{d}\theta,\,\mathrm{d}\phi)\,. \tag{4.123}
$$

The integrals can therefore be replaced with integrals over the boundary of $\widehat{\mathfrak{M}}_{u|\theta\phi}$, which is just the two loci $u=0$ and $y=0$, with opposite orientation. The $\log(y)$ in ν_1 diverges at $y=0$ so we will regulate by evaluating I_1 at $y=\varepsilon$ then taking $\varepsilon\to 0$. The measures need to be expressed in terms of the appropriate bosonic moduli for each boundary component, so for $u\to 0$ we need to write

$$
\nu_1 = -\Big(\log(1-u)+\frac{\theta\phi}{1-u}\Big)\,\delta^2(\mathrm{d}\theta,\,\mathrm{d}\phi)\,. \tag{4.124}
$$

This gives us

$$
\begin{aligned}
I_1 = \int_{y=\varepsilon}\nu_1 - \int_{u=0}\nu_1 &= -\int \log(\varepsilon)\,\delta^2(\mathrm{d}\theta,\,\mathrm{d}\phi) + \int (\log(1)+\theta\phi)\,\delta^2(\mathrm{d}\theta,\,\mathrm{d}\phi) \\
&= -\int \mathrm{d}\theta\,\mathrm{d}\phi\,\log(\varepsilon) + \int \mathrm{d}\theta\,\mathrm{d}\phi\,\theta\phi = -1
\end{aligned}
\tag{4.125}
$$

where in the last line we used the usual Berezin integral. There is no dependence on the cutoff ε. Similarly, to compute I_2 we need to express u as $1-y+\theta\phi$ near $y=0$, and we find

$$
I_2 = \int_{y=0}\nu_1 - \int_{u=0}\nu_1 = \int (1+\theta\phi)\,\delta^2(\mathrm{d}\theta,\,\mathrm{d}\phi)) - 0 \tag{4.126}
$$

$$
= \int \mathrm{d}\theta\,\mathrm{d}\phi\,(1+\theta\phi) = -1\,. \tag{4.127}
$$

Inserting the values for I_1 and I_2 from Eq. (4.125) and Eq. (4.127) into Eq. (4.119), we obtain an expression for the remaining part of the field theory amplitude.

$$\begin{aligned} A_{\mathrm{QFT}}^{\mathrm{rem}} = -\frac{g^2}{(4\pi)^d} \int_0^\infty \Big[\prod_{i=1}^{2} \frac{\mathrm{d}t_i\, \mathrm{e}^{-t_i m_i^2}\, gB_i}{t_i^{d/2-1} \sinh(gB_i t_i)}\Big] & \\ \times \Big\{ \big(d-2+2\cosh(2gB_1t_1)+N_{\mathrm{s}}-2\big) & \\ \times \big(d-2+2\cosh(2gB_2t_2)+N_{\mathrm{s}}-2\big) & \\ - \big(d-2+2\cosh(2gB_1t_1+2gB_2t_2)+N_{\mathrm{s}}\big)\Big\}\,. & \end{aligned} \tag{4.128}$$

We can retrace our steps of the calculation and express this in terms of the various world-sheet CFTs the terms originated from as we did for the non-separating degeneration in Eq. (4.91); we find

$$\begin{aligned} A_{\mathrm{QFT}}^{\mathrm{rem}} = -\frac{g^2}{(4\pi)^d} \int_0^\infty \Big[\prod_{i=1}^{2} \frac{\mathrm{d}t_i\, \mathrm{e}^{-t_i m_i^2}\, gB_i}{t_i^{d/2-1} \sinh(gB_i t_i)}\Big] \Big\{ \mathbf{f}_{\parallel}^{11} + \mathbf{f}_{\perp}^{11} + \mathbf{f}_{\mathrm{scal}}^{11} & \\ + \big(\mathbf{f}_{\parallel}^{10} + \mathbf{f}_{\perp}^{10} + \mathbf{f}_{\mathrm{scal}}^{10} + \mathbf{f}_{\mathrm{gh}}^{10}\big)\big(\mathbf{f}_{\parallel}^{01} + \mathbf{f}_{\perp}^{01} + \mathbf{f}_{\mathrm{scal}}^{01} + \mathbf{f}_{\mathrm{gh}}^{01}\big)\Big\}\,, & \end{aligned} \tag{4.129}$$

where

$$\mathbf{f}_{\parallel}^{10} = 2\cosh(2gB_1t_1) \qquad \mathbf{f}_{\perp}^{10} = d-2 \qquad \mathbf{f}_{\mathrm{scal}}^{10} = N_{\mathrm{s}} \qquad \mathbf{f}_{\mathrm{gh}}^{10} = -2 \tag{4.130}$$

$$\mathbf{f}_{\parallel}^{01} = 2\cosh(2gB_2t_2) \qquad \mathbf{f}_{\perp}^{01} = d-2 \qquad \mathbf{f}_{\mathrm{scal}}^{01} = N_{\mathrm{s}} \qquad \mathbf{f}_{\mathrm{gh}}^{01} = -2 \tag{4.131}$$

$$\mathbf{f}_{\parallel}^{11} = -2\cosh\big(2g(B_1t_1+B_2t_2)\big) \quad \mathbf{f}_{\perp}^{11} = -(d-2) \quad \mathbf{f}_{\mathrm{scal}}^{11} = -N_{\mathrm{s}}\,. \tag{4.132}$$

Note there is no contribution $\mathbf{f}_{\mathrm{gh}}^{11}$; this corresponds to the fact that in the infinite product in $\mathbf{F}_{\mathrm{gh}}(k_i,\eta)$ in Eq. (4.9), n ranges from 2 to ∞, not from 1 to ∞ as in $\mathbf{F}_{\mathrm{gl}}$ and $\mathbf{F}_{\mathrm{scal}}$, and therefore there is no term proportional to $\sqrt{k}(\mathbf{S}_1\mathbf{S}_2)$ in that CFT.

The second line of Eq. (4.129) can be factorized; we can write

$$\begin{aligned} A_{\mathrm{QFT}}^{\mathrm{rem}} = -\frac{g^2}{(4\pi)^d} \int_0^\infty \Big[\prod_{i=1}^{2} \frac{\mathrm{d}t_i\, \mathrm{e}^{-t_i m_i^2}\, gB_i}{t_i^{d/2-1} \sinh(gB_i t_i)}\Big] \Big\{ \mathbf{f}_{\parallel}^{11} + \mathbf{f}_{\perp}^{11} + \mathbf{f}_{\mathrm{scal}}^{11} \Big\} & \\ - g^2 \prod_{i=1}^{2} \Big[\frac{1}{(4\pi)^{d/2}} \int_0^\infty \frac{\mathrm{d}t_i}{t_i^{d/2-1}} \frac{\mathrm{e}^{-t_i m_i^2}\, gB_i}{\sinh(gB_i t_i)} \big(\mathbf{f}_{\parallel}^{i} + \mathbf{f}_{\perp}^{i} + \mathbf{f}_{\mathrm{scal}}^{i} + \mathbf{f}_{\mathrm{gh}}^{i}\big)\Big]\,, & \end{aligned} \tag{4.133}$$

where

$$\mathbf{f}_{\parallel}^{i} = 2\cosh(2gB_it_i) \qquad \mathbf{f}_{\perp}^{i} = d-2 \qquad \mathbf{f}_{\mathrm{scal}}^{i} = N_{\mathrm{s}} \qquad \mathbf{f}_{\mathrm{gh}}^{i} = -2\,. \tag{4.134}$$

4.7 Comparison with bosonic string theory

There are a number of differences between our approach, which uses the NS sector of an superstring, and the approach in previous works [101, 16, 94] which uses bosonic string theory; the differences are also discussed in [1].

The first difference is that the worldsheets are 2-dimensional manifolds, not supermanifolds; these can be parametrized as Riemann surfaces. We are focusing on open string world-sheets with boundaries but no handles; in the simplest case, the relevant world-sheet has the topology of the disk, which can be conformally mapped to $\overline{\mathbf{C}}^+$, the upper-half part of the complex plane plus the point at infinity, with the real line representing the boundary. Higher-genus Riemann surfaces can be constructed with Schottky groups, *i.e.* as the quotient of a subset of $\overline{\mathbf{C}}^+$ by a free group of projective transformations subject to certain conditions. Schottky groups are described in section 2.3.1.

To give an example of the typical form of the expressions for geometric objects in the Schottky parametrization, let us begin by considering the measure of integration for the disk with all three boundaries lying on the same D-brane, so all open strings are uncharged. If we use $\mathrm{SL}(2,\mathbf{R})$ invariance of the amplitude to choose the fixed points of the two Schottky group generators as

$$\eta_1 = 0\,; \qquad \xi_1 = \infty\,; \qquad \eta_2 = \eta\,; \qquad \xi_2 = 1\,; \tag{4.135}$$

and denote their multipliers as k_1, k_2, then the amplitude can be written as

$$[\mathrm{d}m]_2^0 \;=\; \frac{\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}\eta}{k_1^2\,k_2^2\,(1-\eta)^2}\,F_{\mathrm{gh}}(k_i,\eta)\,F_{\mathrm{gl}}^{(0)}(k_i,\eta)\,F_{\mathrm{scal}}(k_i,\eta)\,, \tag{4.136}$$

which is the bosonic string analogue analogue of Eq. (4.4), where again we have labelled the various factors anticipating the role that they are going to play in the field theory limit, as discussed below. Note that the integration variable η is not just one of the four fixed points; it is also equal to the projective-invariant cross ratio $\eta = (\eta_1, \eta_2, \xi_1, \xi_2)$ in the notation of Eq. (2.205). As we did in the NS sector, we factorize $F_{\mathrm{gl}}^{(0)} = F_{\|}(k_i,\eta) F_{\perp}(k_i,\eta)$ into a part in the same plane as the magnetic fields and a perpendicular part. With the appropriate modifications of the measure in [7] as derived in [16], we obtain

$$\begin{aligned}
F_{\mathrm{gh}}(k_i,\eta) &= (1-k_1)^2\,(1-k_2)^2 \prod_{\alpha}{}' \prod_{n=2}^{\infty} (1-k_\alpha^n)^2\,, \\
F_{\|}(k_i,\eta) &= \mathrm{e}^{-\mathrm{i}\pi\vec{\epsilon}\cdot\tau\cdot\vec{\epsilon}}\Big[\det\left(\mathrm{Im}\,\tau_{\vec{\epsilon}}\right)\Big]^{-1} \prod_{\alpha}{}' \prod_{n=1}^{\infty} \left(1-\mathrm{e}^{2\pi\mathrm{i}\vec{\epsilon}\cdot\tau\cdot\vec{N}_\alpha}\,k_\alpha^n\right)^{-1}\left(1-\mathrm{e}^{-2\pi\mathrm{i}\vec{\epsilon}\cdot\tau\cdot\vec{N}_\alpha}\,k_\alpha^n\right)^{-1}, \\
F_{\perp}(k_i,\eta) &= \Big[\det\left(\mathrm{Im}\,\tau\right)\Big]^{-\frac{d-2}{2}} \prod_{\alpha}{}' \prod_{n=1}^{\infty} (1-k_\alpha^n)^{-d+2}\,, \\
F_{\mathrm{scal}}(k_i,\eta) &= \prod_{I=1}^{N_{\mathrm{s}}} \mathrm{e}^{2\pi\mathrm{i}\alpha' \vec{m}_I\cdot\tau\cdot\vec{m}_I} \prod_{\alpha}{}' \prod_{n=1}^{\infty} (1-k_\alpha^n)^{-N_{\mathrm{s}}}\,.
\end{aligned} \tag{4.137}$$

As in the NS sector, the product $\prod_\alpha{}'$ is over all elements $T_\alpha \in \mathcal{S}(2)$ which are not integer powers of other elements, taken modulo cyclic permutations of their factors, and with the identity excluded; τ is the period matrix of the Riemann surface, whose expression in the Schottky parametrization can be found, for instance, in Eq. (A.14) of [8].

$\tau_{\vec{\epsilon}}$ is the twisted period matrix; the bosonic equivalent of $\boldsymbol{\tau}_{\vec{\epsilon}}$; it is explained how to compute $\tau_{\vec{\epsilon}}$ in section 3.2. As in the NS amplitude, $\det\left(\mathrm{Im}\,\tau\right)$ has been replaced with

$\det(\mathrm{Im}\,\tau_{\vec{\epsilon}})$ for the sector with twisted periodicity.

The most obvious difference between the measures in Eq. (4.136) and Eq. (4.4) is the occurrence of square roots of the multipliers as well as integer powers. In the bosonic string, the mass level of states propagating in the μth loop increases with the power of k_μ, while in the superstring the mass level of states propagating in the μth loop increases with the power of $k_\mu{}^{\frac{1}{2}}$. Necessarily, the propagation of a massless state must correspond to a term like $\mathrm{d}k_\mu/k_\mu = \mathrm{d}\log k_\mu$ in the integrand, so tachyons propagating in loops correspond to terms like $\mathrm{d}k_\mu/k_\mu^2$ in the bosonic theory and $\mathrm{d}k_\mu/k_\mu^{3/2}$ in the superstring, which is why terms like these appear in Eq. (4.136) and Eq. (4.4), respectively.

In both theories, we can find the QFT amplitude by expanding the measure in powers of k_μ for the bosonic string or $k_\mu{}^{\frac{1}{2}}$ for the superstring, isolating the term corresponding to massless states propagating in all loops, then taking the $\alpha' \to 0$ limit. An important difference is that while this amounts to an *ad hoc* removal of the tachyonic states by hand in the bosonic theory, this is not necessary with superstrings for they are automatically eliminated from the spectrum upon integrating over the odd moduli and carrying out the GSO projection. In both theories, higher-level states become infinitely massive and decouple in the $\alpha' \to 0$ limit.

To find the QFT diagrams with the topology of Fig. 4.4a, we need to switch to parameters p_i defined by the same equations as in Eq. (4.39) but with k_3 denoting the multiplier of the Schottky group element $S_1^{-1}S_2$ instead of the super-Schottky group element $\mathbf{S}_1^{-1}\mathbf{S}_2$. Then the cross-ratio of the four fixed points becomes

$$\eta = \frac{(1+p_1)(1+p_2)p_3}{(1+p_3)(1+p_1p_2p_3)}\,, \tag{4.138}$$

and the integration measure become symmetric:

$$\frac{\mathrm{d}k_1}{k_1^2}\,\frac{\mathrm{d}k_2}{k_2^2}\,\frac{\mathrm{d}\eta}{(1-\eta)^2}(1-k_1)^2(1-k_2)^2 \;=\; \frac{\mathrm{d}p_1}{p_1^2}\,\frac{\mathrm{d}p_2}{p_2^2}\,\frac{\mathrm{d}p_3}{p_3^2}\,(1-p_2p_3)(1-p_1p_3)(1-p_1p_2)\,. \tag{4.139}$$

While the massless sector of the bosonic string amplitude in the p_i parameters ends up the same as the corresponding amplitude in the NS sector of the superstring, various contributions can arise in slightly different ways in the two approaches. For example, the twisted determinant for the bosonic string $\det(\mathrm{Im}\,\tau_{\vec{\epsilon}})$ is given at lowest order in k_μ by

$$\begin{aligned}\det(\mathrm{Im}\,\tau_{\vec{\epsilon}}) = {}& \frac{1}{4\pi^2}\Gamma(-\epsilon_1)\Gamma(-\epsilon_2)\Gamma(\epsilon_1+\epsilon_2)\Bigg(\epsilon_1\big(k_1^{\frac{\epsilon_1}{2}}\eta^{-\frac{\epsilon_1-\epsilon_2}{2}} - k_1^{-\frac{\epsilon_1}{2}}\eta^{-\frac{\epsilon_1+\epsilon_2}{2}}\big)\\ &\times \big(k_1^{\frac{\epsilon_2}{2}}\eta^{\frac{\epsilon_1-\epsilon_2}{2}} - k_2^{-\frac{\epsilon_2}{2}}\eta^{-\frac{\epsilon_1-\epsilon_2}{2}}\big)\,{}_2F_1(1-\epsilon_1,-\epsilon_2;1-\epsilon_1-\epsilon_2;\eta)\\ &+\Big(\epsilon_2 k_1^{\frac{\epsilon_1}{2}}\eta^{-\frac{\epsilon_1}{2}}\big(k_1^{\frac{\epsilon_2}{2}}\eta^{\frac{\epsilon_1}{2}} - k_2^{-\frac{\epsilon_2}{2}}\eta^{-\frac{\epsilon_1}{2}}\big) + \epsilon_1 k_2^{-\frac{\epsilon_2}{2}}\eta^{-\epsilon_1-\frac{\epsilon_2}{2}}\big(k_1^{\frac{\epsilon_1}{2}}\eta^{\frac{\epsilon_2}{2}} - k_1^{-\frac{\epsilon_1}{2}}\eta^{-\frac{\epsilon_2}{2}}\big)\Big)\\ &\times\,{}_2F_1(-\epsilon_1,\epsilon_2;1-\epsilon_1-\epsilon_2;\eta)\Bigg) + (\epsilon_\mu \leftrightarrow -\epsilon_\mu) + \mathcal{O}(k_\mu)\,. \end{aligned} \tag{4.140}$$

We can expand this to first order in p_3 with the substitutions in Eq. (4.39) and Eq. (4.138), using Eq. (4.54) to expand the hypergeometric functions. Rewriting in terms of the field theory variables with Eq. (4.62) and Eq. (4.63), we get

$$\det(\operatorname{Im}\tau_{\vec{\epsilon}}) = \frac{1}{4\pi^2(\alpha')^2}\Big(\Delta_F - 2\alpha' p_3 \cosh(2gB_3t_3 - gB_1t_1 - gB_2t_2)\frac{\sinh(gB_3t_3)}{gB_3}\Big) + \mathcal{O}(p_1, p_2, p_3^2) + \mathcal{O}(\alpha')\,, \tag{4.141}$$

where Δ_F is defined in Eq. (4.65). The term proportional to p_3 in Eq. (4.141) will end up as one of the factors in a Feynman diagram with a gluon polarized parallel to the magnetic field propagating in the leg t_3. The p_3 term here receives a contribution from the first-order term in the series expansion of the hypergeometric function, Eq. (4.54).

For the superstring, the situation changes: we keep terms only of order $\sqrt{p_i}$, because terms of order p_i and higher will become massive states which decouple. This means that all of the hypergeometric functions in the expression for the supersymmetric twisted determinant, Eq. (3.59), are equivalent to unity. To get the analogue of Eq. (4.140), an expression for the supersymmetric twisted determinant at lowest order in the k_μ, we take the first five lines of Eq. (3.59). If we set $\theta\phi \to 0$ and $u \to \eta$, then this is identical to Eq. (4.140), *i.e.*

$$\det(\operatorname{Im}\boldsymbol{\tau}_{\vec{\epsilon}}) = \det(\operatorname{Im}\tau_{\vec{\epsilon}})\Big|_{\eta=u} + \mathcal{O}(\theta\phi) + \mathcal{O}(k_\mu)\,. \tag{4.142}$$

To check this requires relations between contiguous hypergeometric functions (§14.7 of [102]) which are not included in Mathematica.

Since the first-order term in the expansion of the hypergeometric functions is crucial to getting the correct coefficient of p_3 in Eq. (4.141), which is important to obtain matching with QFT diagrams, it is necessary that the $\mathcal{O}(\theta\phi)$ term in Eq. (4.142) will compensate the fact that the hypergeometric functions are equivalent to unity at order $\sqrt{p_3}$.

This is exactly the situation. The $\mathcal{O}(\theta\phi)$ term in Eq. (4.142), *i.e.* the fourth and fifth lines of $\det(\operatorname{Im}\boldsymbol{\tau}_{\vec{\epsilon}})$ in Eq. (3.59), has the right form that when the variables u, k_μ are rewritten in terms of p_i using Eq. (4.39) and the nilpotent object $\theta\phi$ is rewritten in terms of $\hat{\theta}_{12}\hat{\phi}_{12}$ using Eq. (4.47), we obtain an expression for $\det(\operatorname{Im}\boldsymbol{\tau}_{\vec{\epsilon}})$ which is almost exactly the same as the corresponding expression for $\det(\operatorname{Im}\tau_{\vec{\epsilon}})$ at lowest order in p_1 and p_2, but with the replacement $p_3 \to \sqrt{p_3}\hat{\theta}_{12}\hat{\phi}_{12}$. In terms of the field theory variables we have

$$\det(\operatorname{Im}\boldsymbol{\tau}_{\vec{\epsilon}}) = \frac{1}{4\pi^2(\alpha')^2}\Big(\Delta_F - 2\alpha'\sqrt{p_3}\hat{\theta}_{12}\hat{\phi}_{12}\cosh(2gB_3t_3 - gB_1t_1 - gB_2t_2)\frac{\sinh(gB_3t_3)}{gB_3}\Big) + \mathcal{O}(\sqrt{p_1}, \sqrt{p_2}, p_3) + \mathcal{O}(\alpha')\,. \tag{4.143}$$

Note that the p_3 and $\sqrt{p_3}\,\hat{\theta}_{12}\hat{\phi}_{12}$ terms in $\det(\operatorname{Im}\tau_{\vec{\epsilon}})$ and $\det(\operatorname{Im}\boldsymbol{\tau}_{\vec{\epsilon}})$, respectively, also receive contributions from sources other than the ones we have discussed, namely factors like $\eta^{n_i\epsilon_\mu/2}$ and $u^{n_i\epsilon_\mu/2}$, respectively. But it is easy to see that these contribute to both sides of Eq. (4.142) in the same way, since we have

$$\eta^{n_i\epsilon_\mu/2} = p_3^{n_i\epsilon_\mu/2}\Big(1 + \frac{n_i\epsilon_\mu}{2}p_3\Big) + \mathcal{O}(p_1, p_2, p_3^2) \tag{4.144}$$

$$u^{n_i\epsilon_\mu/2} = p_3^{n_i\epsilon_\mu/2}\Big(1 + \frac{n_i\epsilon_\mu}{2}\sqrt{p_3}\,\hat{\theta}_{12}\hat{\phi}_{12}\Big) + \mathcal{O}(\sqrt{p_1}, \sqrt{p_2}, p_3)\,. \tag{4.145}$$

There is a similar relationship between $\det(\mathrm{Im}\,\tau_{\vec{\epsilon}})$ and $\det(\mathrm{Im}\,\boldsymbol{\tau}_{\vec{\epsilon}})$. So as required, when all of the other factors are inserted, the coefficient of $p_1p_2p_3$ in the bosonic string measure is the same as the coefficient of $\mathrm{e}^{\mathrm{i}\pi(\varsigma_1+\varsigma_2)}\sqrt{p_1p_2p_3}\,\hat{\theta}_{12}\hat{\phi}_{12}$ in the superstring measure, and the same QFT amplitude is obtained for the massless sectors of the bosonic and supersymmetric theories.

The terms computed in section 4.6, which vanish in the p_i parametrization and correspond to QFT diagrams with the topology of Fig. 4.4b and Fig. 4.4c also appear in the bosonic theory, where they similarly disappear when the p_i parameters are used. We get once more Eq. (4.119) for the QFT limit, but with I_1 and I_2 now representing the integrals

$$I_1 = \int_0^1 \frac{\mathrm{d}\eta}{(1-\eta)^2}\,, \qquad\qquad I_2 = -\int_0^1 \mathrm{d}\eta. \tag{4.146}$$

In the superstring case we used two different super-projective cross-ratios u and y to compute the integral over the third bosonic modulus. In this case, however, any projective cross-ratio of four points can be related in a simple way to any other projective cross-ratio of the same four points (the analogous statement does not hold for super-projective cross-ratios, see *e.g.* Eq. (2.220)), so we stick with η and integrate it between 0 and 1.

Clearly, the bosonic $I_2 = -1$ as in Eq. (4.120). The bosonic I_1, on the other hand, clearly diverges without regularization. A number of arguments are given in section 5 of [103] that integrals similar to this (for diagrams with external momenta) should be set to $I_2 = -\frac{1}{2}$. For example, expanding the integrand as a series in η and integrating term by term gives $1 + 1 + \ldots = -\frac{1}{2}$ using ζ-function regularization arguments. In particular, this result gives the correct QFT limit in that case. This differs from our calculation of I_1 for the superstring in Eq. (4.125) by a factor of $\frac{1}{2}$. This means that while our calculation of this term in the superstring case was finite and didn't need regularization unlike the bosonic case, it has the wrong factor to provide a 1–1 match with Feynman diagrams with the relevant topology (*e.g.* Fig. 4.4b), for which $I_1 = -\frac{1}{2}$ is the result required for diagram-by-diagram matching between QFT and string theory. The discrepancy may arise from an assumption that only states from the massless sector can propagate through the separating edge, following from momentum conservation, but perhaps the string theory result in fact includes contributions from higher mass levels which wouldn't appear in the QFT analysis of the massless sector. It is also possible that the discrepancy arises from an incorrect choice of bosonic integration variable for this topology, and that a careful analysis of the sewing procedure will lead to the correct result.

Chapter 5

Yang-Mills theory in background Gervais-Neveu gauge

In this chapter we investigate vacuum diagrams for a U(N) Yang-Mills gauge field minimally coupled to an adjoint scalar in a covariantly constant background field. We use an appropriate modification of the non-linear gauge introduced by Gervais and Neveu [11] to match diagrams with terms obtained from the $\alpha' \to 0$ limit of string theory. We also give a vacuum expectation value (VEV) to the scalar field (breaking the gauge symmetry) so it will act as an IR regulator.

5.1 The Lagrangian

The Lagrangian is obtained by dimensionally reducing the pure Yang-Mills Lagrangian from D dimensions to $d < D$ dimensions. We split the gauge field $\mathcal{A}_M$ up into a fixed classical background A_M and a quantum field Q_M: $\mathcal{A}_M = A_M + \mathcal{Q}_M$, where $M = 0, \ldots, D-1$. The field strength $\mathcal{F}_{MN}$ can be expressed in terms of the covariant derivative $\mathscr{D}^{\mathcal{A}}_M = \partial_M + \mathrm{i}\, g[\mathcal{A}_M, \cdot]$ as

$$\mathcal{F}_{MN} = \frac{1}{\mathrm{i}\, g}[\mathscr{D}^{\mathcal{A}}_M, \mathscr{D}^{\mathcal{A}}_N] = F_{MN} + \mathscr{D}_M \mathcal{Q}_N - \mathscr{D}_N \mathcal{Q}_M + \mathrm{i}\, g[\mathcal{Q}_M, \mathcal{Q}_N]\,, \tag{5.1}$$

where we have split the covariant derivative as

$$\mathscr{D}^{\mathcal{A}}_M = \partial_M + \mathrm{i}\, g[A_M, \cdot] + \mathrm{i}\, g[\mathcal{Q}_M, \cdot] \equiv \mathscr{D}_M + \mathrm{i}\, g[\mathcal{Q}_M, \cdot]\,, \tag{5.2}$$

i.e. $\mathscr{D}_M$ (without an $\mathcal{A}$) denotes the covariant derivative with respect to the background gauge field. $F_{MN} = \partial_M A_N - \partial_N A_M$ is the background field strength. The classical Lagrangian is given, then, by

$$\begin{aligned}\mathcal{L}_{\mathrm{cl}} &= -\frac{1}{2}\mathrm{Tr}(\mathcal{F}_{MN}\mathcal{F}^{MN}) &(5.3)\\ &= \mathrm{Tr}\Big(-\frac{1}{2}F_{MN}F^{MN} + \mathscr{D}^{\mu}Q^{\nu}\mathscr{D}_N\mathcal{Q}_M - \mathscr{D}^M Q^N \mathscr{D}_M \mathcal{Q}_N + 2\mathrm{i}\, g Q^M F_{MN} Q^N \\ &\qquad - 2\mathrm{i}\, g\mathscr{D}^M \mathcal{Q}^N[\mathcal{Q}_M, Q_N] + \frac{1}{2}g^2[\mathcal{Q}_M, \mathcal{Q}_N][\mathcal{Q}_M, \mathcal{Q}_N]\Big) &(5.4)\end{aligned}$$

where $\mathrm{Tr}(\cdot)$ here denotes the trace over the $\mathfrak{u}(N)$ Lie algebra. We have removed a term linear in Q_M because it can be absorbed into a redefinition of the current.

The gauge condition we want to impose is

$$G = \mathscr{D}_M \mathcal{Q}^M + \mathrm{i}\,\gamma g \mathcal{Q}_M \mathcal{Q}^M = 0\,, \tag{5.5}$$

where γ is a gauge parameter, so the gauge-fixing Lagrangian $\mathcal{L}_{\mathrm{gf}}$ is given as

$$\begin{aligned}\mathcal{L}_{\mathrm{gf}} &= -\mathrm{Tr}\Big(\big(\mathscr{D}_M \mathcal{Q}^M + \mathrm{i}\,\gamma g \mathcal{Q}_M \mathcal{Q}^M\big)^2\Big) && (5.6)\\ &= -\mathrm{Tr}\big(\mathscr{D}_M \mathcal{Q}^M \mathscr{D}_N \mathcal{Q}^N + 2\mathrm{i}\,\gamma g \mathscr{D}_M \mathcal{Q}^M \mathcal{Q}_N \mathcal{Q}^N - \gamma^2 g^2 \mathcal{Q}_M \mathcal{Q}^M \mathcal{Q}_N \mathcal{Q}^N\big). && (5.7)\end{aligned}$$

Using partial integration and the cyclic property of $\mathrm{Tr}(.)$, we can combine $\mathcal{L}_{\mathrm{cl}}$ from Eq. (5.4) and $\mathcal{L}_{\mathrm{gf}}$ from Eq. (5.7) as

$$\begin{aligned}\mathcal{L}_{\mathrm{cl}} + \mathcal{L}_{\mathrm{gf}} = \mathcal{L}_{\mathrm{bf}} + \mathrm{Tr}\big(&\mathcal{Q}_M \mathscr{D}_N \mathscr{D}^N \mathcal{Q}^M + 4\mathrm{i}\,g \mathcal{Q}^M F_{MN} \mathcal{Q}^N + 2\mathrm{i}\,g\gamma \mathscr{D}_M \mathcal{Q}^M \mathcal{Q}_N \mathcal{Q}^N \\ &+ 2\mathrm{i}\,g \mathscr{D}_M \mathcal{Q}_N [\mathcal{Q}^M, \mathcal{Q}^N] + g^2(\eta_{RM}\eta_{SN} + (\gamma^2 - 1)\eta_{RN}\eta_{SM}) \mathcal{Q}^M \mathcal{Q}^N \mathcal{Q}^R \mathcal{Q}^S\big)\end{aligned} \tag{5.8}$$

where we've separated the classical Lagrangian for the background field, $\mathcal{L}_{\mathrm{bf}} = -\frac{1}{2}\mathrm{Tr}(F_{MN}F^{MN})$.

Lastly, we need the Lagrangian for the Faddeev-Popov ghost fields $\overline{\mathcal{C}}, \mathcal{C}$, which can be found by computing the variation $\delta_\theta G$ with respect to a gauge transformation of the gauge condition Eq. (5.5), then making the replacement $\delta_\theta \mathcal{Q}_M \to \theta \mathscr{D}_M^A \mathcal{C}$, and inserting this into $\mathrm{Tr}[\overline{c}, \cdot]$, which gives

$$\mathcal{L}_{\mathrm{gh}} = 2\mathrm{Tr}\big(- \overline{\mathcal{C}} \mathscr{D}_M \mathscr{D}^M \mathcal{C} + \mathrm{i}\,g \mathscr{D}_M \overline{\mathcal{C}} [\mathcal{Q}^M, \mathcal{C}] - \mathrm{i}\,\gamma g\, \overline{\mathcal{C}} \{Q_M, \mathscr{D}^M \mathcal{C}\} + \gamma g^2 \overline{\mathcal{C}} \{\mathcal{Q}_M, [\mathcal{Q}^M, \mathcal{C}]\}\big)\,. \tag{5.9}$$

Now we have the full pure Yang-Mills lagrangian in D dimensions; we want to dimensionally reduce to d dimensions. The D-dimensional gauge field splits into a d-dimensional gauge field and $N_{\mathrm{s}} \equiv D - d$ adjoint scalars:

$$(\mathcal{Q}_M) \to (\mathcal{Q}_\mu; \Phi_I) \tag{5.10}$$

for $\mu = 0, \ldots, d-1$; $I = 1, \ldots, N_{\mathrm{s}}$; while the covariant derivative splits into a d-dimensional covariant derivative and a commutator with the background scalar fields, which we will give VEVs to:

$$(\mathscr{D}_M) \to (\mathscr{D}_\mu \equiv \partial_\mu + \mathrm{i}\,g[A_\mu, \cdot];\ \mathrm{i}\,[\mathscr{M}_I, \cdot])\,. \tag{5.11}$$

Note that the D-dimensional d'Alembertian splits into a d-dimensional d'Alembertian plus a mass term:

$$\mathscr{D}_M \mathscr{D}^M X = \mathscr{D}_\mu \mathscr{D}^\mu X + [\mathscr{M}_I, [\mathscr{M}_I, X]]\,, \tag{5.12}$$

where for the reduced dimensions, our summation notation does not include the negative signature of the metric, and just means a summation over the flavour indices $A_I B_I =$

$\sum_{I=1}^{N_s} A_I B_I$. The dimensionally-reduced Lagrangian can be written as the sum of the following terms:

$$\mathcal{L}_{\mathcal{Q}^2} = \mathrm{Tr}[\mathcal{Q}^\mu(\mathscr{D}_\nu\mathscr{D}^\nu\mathcal{Q}_\mu + 4\mathrm{i}\,gF_{\mu\rho}\mathcal{Q}^\rho + [\mathscr{M}_I,[\mathscr{M}_I,\mathcal{Q}_\mu]])] \tag{5.13}$$
$$\mathcal{L}_{\Phi^2} = \mathrm{Tr}[-\Phi_I\mathscr{D}_\nu\mathscr{D}^\nu\Phi_I - \Phi_I[\mathscr{M}_J,[\mathscr{M}_J,\Phi_I]]] \tag{5.14}$$
$$\mathcal{L}_{\overline{\mathcal{C}}\mathcal{C}} = \mathrm{Tr}[-2\overline{\mathcal{C}}\mathscr{D}_\mu\mathscr{D}^\mu\mathcal{C} - 2\overline{\mathcal{C}}[\mathscr{M}_J,[\mathscr{M}_J,\mathcal{C}]]] \tag{5.15}$$
$$\mathcal{L}_{\mathcal{Q}^3} = -2\mathrm{i}\,g\gamma\mathrm{Tr}[\mathscr{D}_\mu\mathcal{Q}^\mu\mathcal{Q}_\nu\mathcal{Q}^\nu] - 2\mathrm{i}\,g\mathrm{Tr}[\mathscr{D}_\mu\mathcal{Q}_\nu[\mathcal{Q}^\mu,\mathcal{Q}^\nu]] \tag{5.16}$$
$$\mathcal{L}_{\mathcal{Q}\Phi^2} = 2\mathrm{i}\,g\gamma\mathrm{Tr}[\mathscr{D}_\mu\mathcal{Q}^\mu\Phi_I\Phi_I] + 2\mathrm{i}\,g\mathrm{Tr}[\mathscr{D}_\mu\Phi_I[\mathcal{Q}^\mu,\Phi_I]] \tag{5.17}$$
$$\mathcal{L}_{\overline{\mathcal{C}}\mathcal{C}\mathcal{Q}} = 2\mathrm{i}\,g\mathrm{Tr}[\mathscr{D}_\mu\overline{\mathcal{C}}[\mathcal{Q}^\mu,\mathcal{C}]] - 2\mathrm{i}\,\gamma g\mathrm{Tr}[\overline{\mathcal{C}}\{\mathcal{Q}_\mu,\mathscr{D}^\mu\mathcal{C}\}] \tag{5.18}$$
$$\mathcal{L}_{\Phi\mathcal{Q}^2} = -2\gamma g\mathrm{Tr}[[\mathscr{M}_I,\Phi_I]Q_\mu Q^\mu] - 2g\mathrm{Tr}[[\mathscr{M}_I,\mathcal{Q}_\mu][\Phi_I,\mathcal{Q}^\mu]] \tag{5.19}$$
$$\mathcal{L}_{\Phi^3} = 2\gamma g[[\mathscr{M}_I,\Phi_I]\Phi_J\Phi_J] + 2g\mathrm{Tr}[[\mathscr{M}_I,\Phi_J][\Phi_I,\Phi_J]] \tag{5.20}$$
$$\mathcal{L}_{\Phi\overline{\mathcal{C}}\mathcal{C}} = 2g\mathrm{Tr}[[\mathscr{M}_I,\overline{\mathcal{C}}][\Phi_I,\mathcal{C}]] - 2\gamma g\mathrm{Tr}[\overline{\mathcal{C}}\{\Phi_I,[M_I,\mathcal{C}]\}] \tag{5.21}$$
$$\mathcal{L}_{\mathcal{Q}^4} = g^2(\eta_{\rho\mu}\eta_{\nu\sigma} + (\gamma^2-1)\eta_{\rho\nu}\eta_{\sigma\mu})\mathrm{Tr}[\mathcal{Q}^\mu\mathcal{Q}^\nu\mathcal{Q}^\rho\mathcal{Q}^\sigma] \tag{5.22}$$
$$\mathcal{L}_{\mathcal{Q}^2\Phi^2} = -2g^2\mathrm{Tr}[\Phi_I\mathcal{Q}^\mu\Phi_I\mathcal{Q}_\mu] - 2(\gamma^2-1)g^2\mathrm{Tr}[\Phi_I\Phi_I\mathcal{Q}^\mu\mathcal{Q}_\mu] \tag{5.23}$$
$$\mathcal{L}_{\Phi^4} = g^2\mathrm{Tr}[\Phi_I\Phi_J\Phi_I\Phi_J] + (\gamma^2-1)g^2\mathrm{Tr}[\Phi_I\Phi_I\Phi_J\Phi_J] \tag{5.24}$$
$$\mathcal{L}_{\overline{\mathcal{C}}\mathcal{C}\mathcal{Q}^2} = 2\gamma g^2\mathrm{Tr}[\overline{\mathcal{C}}\{\mathcal{Q}_\mu,[\mathcal{Q}^\mu,\mathcal{C}]\}] \tag{5.25}$$
$$\mathcal{L}_{\overline{\mathcal{C}}\mathcal{C}\Phi^2} = -2\gamma g^2\mathrm{Tr}[\overline{\mathcal{C}}\{\Phi_I,[\Phi_I,\mathcal{C}]\}]\,. \tag{5.26}$$

The gauge condition Eq. (5.5) has become

$$\tilde{G} \equiv \mathscr{D}_\mu\mathcal{Q}^\mu + \mathrm{i}\,\gamma g\mathcal{Q}_\mu\mathcal{Q}^\mu - \mathrm{i}\,[\mathscr{M}_I,\Phi_I] - \mathrm{i}\,\gamma g\Phi_I\Phi_I = 0\,. \tag{5.27}$$

5.2 The Lagrangian in component form

Now, let us assume that A_μ and $\mathscr{M}_I$ all commute and pick a basis of $\mathfrak{u}(N)$ in which they are diagonal. In this basis, let us write

$$\mathscr{M}_I = \begin{pmatrix} m_I^1 & 0 & \dots & 0 \\ 0 & m_I^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & m_I^N \end{pmatrix} \qquad A_\mu = \begin{pmatrix} A_\mu^1 & 0 & \dots & 0 \\ 0 & A_\mu^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_\mu^N \end{pmatrix} \tag{5.28}$$

$$\mathcal{Q}_\mu = \frac{1}{\sqrt{2}}\begin{pmatrix} Q_\mu^{11} & \dots & Q_\mu^{1N} \\ \vdots & \ddots & \vdots \\ Q_\mu^{N1} & \dots & Q_\mu^{NN} \end{pmatrix} \qquad \Phi_I = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_I^{11} & \dots & \phi_I^{1N} \\ \vdots & \ddots & \vdots \\ \phi_I^{N1} & \dots & \phi_I^{NN} \end{pmatrix} \tag{5.29}$$

$$\overline{\mathcal{C}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \overline{c}^{11} & \dots & \overline{c}^{1N} \\ \vdots & \ddots & \vdots \\ \overline{c}^{N1} & \dots & \overline{c}^{NN} \end{pmatrix} \qquad \mathcal{C} = \frac{1}{\sqrt{2}}\begin{pmatrix} c^{11} & \dots & c^{1N} \\ \vdots & \ddots & \vdots \\ c^{N1} & \dots & c^{NN} \end{pmatrix}\,. \tag{5.30}$$

all satisfying $X^{ij} = (X^{ji})^*$ since $\mathfrak{u}(N)$ matrices are Hermitian. Notice that the covariant derivative does not mix between the entries of a matrix:

$$\mathscr{D}_\mu(X^{ij}) = \begin{pmatrix} \partial_\mu X^{11} & (\partial_\mu + \mathrm{i}\, g A_\mu^{12}) X^{12} & \dots & (\partial_\mu + \mathrm{i}\, g A_\mu^{1N}) X^{1N} \\ (\partial_\mu + \mathrm{i}\, g A_\mu^{21}) X^{21} & \partial_\mu X^{22} & \dots & (\partial_\mu + \mathrm{i}\, g A_\mu^{2N}) X^{2N} \\ \vdots & \vdots & \ddots & \vdots \\ (\partial_\mu + \mathrm{i}\, g A_\mu^{N1}) X^{N1} & (\partial_\mu + \mathrm{i}\, g A_\mu^{N2}) X^{N2} & \dots & \partial_\mu X^{NN} \end{pmatrix} \tag{5.31}$$

where we've defined

$$A_\mu^{ij} \equiv A_\mu^i - A_\mu^j\,. \tag{5.32}$$

Motivated by this, we can define a covariant derivative D_μ which acts on matrix entries, not on $\mathfrak{u}(N)$ elements, by

$$X^{ij} \mapsto D_\mu X^{ij} = \partial_\mu + \mathrm{i}\, g A_\mu^{ij} X^{ij}\,; \qquad \mathscr{D}_\mu(X^{ij}) = (D_\mu X^{ij})\,, \tag{5.33}$$

using the notation in which (X^{ij}) denotes the matrix whose (ij)th entry is X^{ij}. D_μ is a derivation if it acts on a product whose colour indices are contracted:

$$D_\mu (XY)^{ij} = (\partial_\mu + \mathrm{i} g A_\mu^{ij}) X^{ik} Y^{kj} = \partial_\mu (X^{ik} Y^{kj}) + \mathrm{i} g (A_\mu^{ik} + A_\mu^{kj}) X^{ik} Y^{kj} \tag{5.34}$$

$$= D_\mu X^{ik} Y^{kj} + X^{ik} D_\mu Y^{kj}\,, \tag{5.35}$$

and so we can partially integrate D_μ in any integrand with contracted colour indices. Similarly, if we define

$$m_I^{ij} \equiv m_I^i - m_I^j\,, \qquad m_{ij}^2 = \sum_{I=1}^{D-d} (m_I^{ij})^2\,, \tag{5.36}$$

then we have

$$[\mathscr{M}_I, (X^{ij})] = \big(m^{ij} X^{ij}\big)\,, \qquad [\mathscr{M}_I, [\mathscr{M}_I, (X^{ij})]] = \big(m_{ij}^2 X^{ij}\big)\,. \tag{5.37}$$

The factors of $\frac{1}{\sqrt{2}}$ in Eq. (5.29) and Eq. (5.30) are necessary so the fields are canonically normalized in the Lagrangian: for example, the term quadratic in Φ in Eq. (5.14) is given by

$$\mathcal{L}_{\Phi^2} = -\frac{1}{2} \phi_I^{ij} D_\mu D^\mu \phi^{ji} - \frac{1}{2} \phi_I^{ij} m_{ij}^2 \phi_I^{ji} \tag{5.38}$$

$$= -\sum_{i=1}^{N} \frac{1}{2} \phi_I^{ii} \partial_\mu \partial^\mu \phi_I^{ii} - \sum_{1 \le i < j}^{N} \Big((\phi_I^{ij})^* D_\mu D^\mu \phi^{ij} + m_{ij}^2 |\phi_I^{ij}|^2 \Big)\,, \tag{5.39}$$

which is the correctly normalized Lagrangian for N massless real scalar fields ϕ^{ii} and $\frac{1}{2}N(N-1)$ complex scalars ϕ^{ij}, $i < j$, with mass $|m_{ij}|$. In terms of these fields, the

Lagrangian is the sum of the following terms:

$$\mathcal{L}_{QQ} = \frac{1}{2}Q^{\mu,ij}D_\nu D^\nu Q^{ji}_\mu + \mathrm{i}\, gQ^{\mu,ij}F^{ji}_{\mu\rho}Q^{\rho,ji} + \frac{1}{2}m^2_{ij}Q^{\mu,ij}Q^{ji}_\mu \tag{5.40}$$

$$\mathcal{L}_{\phi\phi} = -\frac{1}{2}\phi^{ij}_I D_\nu D^\nu \phi^{ji}_I - \frac{1}{2}m^2_{ij}\phi^{ij}_I\phi^{ji}_I \tag{5.41}$$

$$\mathcal{L}_{\overline{c}c} = -\overline{c}^{ij}D_\mu D^\mu c^{ji} - m^2_{ij}\overline{c}^{ij}c^{ji} \tag{5.42}$$

$$\mathcal{L}_{Q^3} = -\frac{\mathrm{i}\, g\gamma}{\sqrt{2}}D_\mu Q^{\mu,ij}Q^{jk}_\nu Q^{\nu,ki} - \frac{\mathrm{i}\, g}{\sqrt{2}}D_\mu Q^{ij}_\nu\left(Q^{\mu,jk}Q^{\nu,ki} - Q^{\nu,jk}Q^{\mu,ki}\right) \tag{5.43}$$

$$\mathcal{L}_{Q\phi^2} = \frac{\mathrm{i}\, g\gamma}{\sqrt{2}}D_\mu Q^{\mu,ij}\phi^{jk}_I\phi^{ki}_I + \frac{\mathrm{i}\, g}{\sqrt{2}}D_\mu\phi^{ij}_I\left(Q^{\mu,jk}\phi^{ki}_I - \phi^{jk}_I Q^{\mu,ki}\right) \tag{5.44}$$

$$\mathcal{L}_{\overline{c}cQ} = \frac{\mathrm{i}g}{\sqrt{2}}D_\mu\overline{c}^{ij}\left(Q^{\mu,jk}c^{ki} - c^{jk}Q^{\mu,ki}\right) - \frac{\mathrm{i}\,\gamma g}{\sqrt{2}}\overline{c}^{ij}\left(Q^{jk}_\mu D^\mu c^{ki} + D^\mu c^{jk}Q^{ki}_\mu\right) \tag{5.45}$$

$$\mathcal{L}_{Q^2\phi} = -\frac{g\gamma}{\sqrt{2}}m^{ij}_I\phi^{ij}Q^{jk}_\mu Q^{\mu,ki} - \frac{g}{\sqrt{2}}m^{ij}_I Q^{ij}_\mu\left(\phi^{jk}_I Q^{\mu,ki} - Q^{\mu,jk}\phi^{ki}_I\right) \tag{5.46}$$

$$\mathcal{L}_{\phi^3} = \frac{\gamma g}{\sqrt{2}}m^{ij}_I\phi^{ij}_I\phi^{jk}_J\phi^{ki}_J + \frac{g}{\sqrt{2}}m^{ij}_I\phi^{ij}_J\left(\phi^{jk}_I\phi^{ki}_J - \phi^{jk}_J\phi^{ki}_I\right) \tag{5.47}$$

$$\mathcal{L}_{\phi\overline{c}c} = \frac{g}{\sqrt{2}}m^{ij}_I\overline{c}^{ij}\left(\phi^{jk}_I c^{jk} - c^{jk}\phi^{ki}_I\right) - \frac{\gamma g}{\sqrt{2}}\overline{c}^{ij}\left(\phi^{jk}_I m^{ki}_I c^{ki} + m^{jk}_I c^{jk}\phi^{ki}_I\right) \tag{5.48}$$

$$\mathcal{L}_{Q^4} = \frac{g^2}{4}(\eta^{\rho\mu}\eta^{\sigma\nu} + (\gamma^2 - 1)\eta^{\rho\nu}\eta^{\sigma\nu})Q^{ij}_\mu Q^{jk}_\nu Q^{k\ell}_\rho Q^{\ell i}_\sigma \tag{5.49}$$

$$\mathcal{L}_{Q^2\phi^2} = -\frac{g^2}{2}\phi^{ij}_I Q^{\mu,jk}\phi^{k\ell}_I Q^{\ell i}_\mu + \frac{1-\gamma^2}{2}g^2\phi^{ij}_I\phi^{jk}_I Q^{\mu,k\ell}Q^{\ell i}_\mu \tag{5.50}$$

$$\mathcal{L}_{\phi^4} = \frac{g^2}{4}\phi^{ij}_I\phi^{jk}_J\phi^{k\ell}_I\phi^{\ell i}_J - \frac{1-\gamma^2}{4}g^2\phi^{ij}_I\phi^{jk}_I\phi^{k\ell}_J\phi^{\ell i}_J \tag{5.51}$$

$$\mathcal{L}_{Q^2\overline{c}c} = \frac{\gamma g^2}{2}\overline{c}^{ij}\left(Q^{jk}_\mu(Q^{\mu,k\ell}c^{\ell i} - c^{k\ell}Q^{\mu,\ell i}) + (Q^{\mu,jk}c^{k\ell} - c^{kj}Q^{\mu,k\ell})Q^{\ell i}_\mu\right) \tag{5.52}$$

$$\mathcal{L}_{\phi^2\overline{c}c} = -\frac{\gamma g^2}{2}\overline{c}^{ij}\left(\phi^{jk}_I(\phi^{k\ell}_I c^{\ell i} - c^{k\ell}\phi^{\ell i}_I) + (\phi^{jk}_I c^{k\ell} - c^{jk}\phi^{k\ell}_I)\phi^{\ell i}_I\right). \tag{5.53}$$

All $\mathfrak{u}(N)$ indices i, j, k, ℓ are to be summed over. $F^{ji}_{\mu\rho}$ in equation Eq. (5.40) is given by

$$F^{ji}_{\mu\rho} = F^{j}_{\mu\rho} - F^{i}_{\mu\rho} \qquad\qquad F^{i}_{\mu\rho} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu\,. \tag{5.54}$$

The quadratic part of this Lagrangian is exactly the same as $\frac{1}{2}N(N-1)$ copies of the quadratic part of the Lagrangian for charged fields ϕ^{ij}, Q^{ij}_μ $\overline{c}^{ij}$, c^{ij}, with mass $|m_{ij}|$ minimally coupled to a U(1) background gauge field $A^{ij}_\mu(x)$, plus N copies of a set of real, massless fields ϕ^{ii}, Q^{ii}_μ $\overline{c}^{ii}$, c^{ii}. The gauge condition Eq. (5.27) has become

$$D^\mu Q^{ij}_\mu + \mathrm{i}\,\gamma g\, Q^{ik}_\mu Q^{\mu,kj} - \mathrm{i}\, m^{ij}_I\phi^{ij}_I - \mathrm{i}\,\gamma g\,\phi^{ik}_I\phi^{ki}_I\,, \tag{5.55}$$

where k is summed over but there is no summation over i or j.

5.3 A constant background field

Now let's specify the form of the background gauge field; let's choose it so for each i, $F^i_{\mu\nu}$ is a $U(1)$ magnetic field in the 1–2 plane. One such gauge field is given by

$$A^i_\mu(x) = x_1 \eta_{\mu 2} B^i \qquad\qquad F^i_{\mu\nu} = (\eta_{\mu 1}\eta_{\nu 2} - \eta_{\nu 1}\eta_{\mu 2}) B^i = \mathcal{A}_{\mu\nu} B^i \,, \tag{5.56}$$

where we've defined the antisymmetric tensor

$$\mathcal{A}_{\mu\nu} = \eta_{\mu 1}\eta_{\nu 2} - \eta_{\nu 1}\eta_{\mu 2} \,. \tag{5.57}$$

The classical action for this background field is given by

$$\mathcal{L}_{\mathrm{bf}} = -\sum_{i=1}^{N} (B^i)^2 = -\frac{1}{N}\Big(\sum_{i=1}^{N} B^i\Big)^2 - \frac{1}{N}\sum_{1\le i<j}^{N} (B^{ij})^2 \,; \qquad B^{ij} \equiv B^i - B^j \,. \tag{5.58}$$

The tree level propagators for the charged component fields ϕ^{ij}, Q^{ij}_μ $\bar{c}^{ij}$, c^{ij} will be the same as the propagators for charged fields of the same type in a U(1) background gauge field $A^{ij}_\mu(x) = x_1\eta_{\mu 2}B^{ij}$. These can be written down exactly in B^{ij}; the expressions are given in the following section.

5.4 Propagators in a constant background field

5.4.1 The scalar propagator

One of the components ϕ^{ij} of the scalar field therefore behaves as a complex scalar charged under the background field

$$A^{ij}_\mu(x) = x_1 \eta_{\mu 2} B^{ij} \,. \tag{5.59}$$

The propagator for scalar fields in this background is calculated in appendix B of [16]. G is given in terms of a heat kernel by

$$G^{ij}(x,y) = \int_0^\infty \mathrm{d}t\, \mathcal{K}^{ij}(x,y;t) \tag{5.60}$$

$$\mathcal{K}^{ij}(x,y;t) = \frac{\mathrm{e}^{-\frac{\mathrm{i}}{2} g B^{ij}(x_1+y_1)(x_2-y_2) - t m^2_{ij}}}{(4\pi t)^{\frac{d}{2}}} \frac{g B^{ij} t}{\sinh(g B^{ij} t)} \times \exp\left[\frac{1}{4}(x_\mu - y_\mu)\beta(F^{ij}, t)^{\mu\nu}(x_\nu - y_\nu)\right] \tag{5.61}$$

where

$$\beta(F^{ij}, t)^{\mu\nu} = \frac{1}{t}\,\eta^{\mu\nu}_{\parallel} + \frac{g B^{ij}}{\tanh(g B^{ij} t)}\,\eta^{\mu\nu}_{\perp} \,. \tag{5.62}$$

The tensors $\eta_{\parallel}^{\mu\nu}$ and $\eta_{\perp}^{\mu\nu}$ project components parallel and perpendicular, respectively, to the background field:

$$\eta_{\parallel}^{\mu\nu} = \mathcal{A}^{\mu\rho}\mathcal{A}_{\rho}{}^{\nu} = \eta^{\mu 1}\delta_2^{\nu} + \eta^{\mu 2}\delta_1^{\nu} \qquad\qquad \eta_{\perp}^{\mu\nu} = \eta^{\mu\nu} - \eta_{\parallel}^{\mu\nu} \,. \tag{5.63}$$

This $G^{ij}(x,y)$ satisfies

$$\Big(\frac{D}{Dx_\mu}\frac{D}{Dx^\mu} + m_{ij}^2\Big)G^{ij}(x,y) = -\mathrm{i}\,\delta^d(x-y)\,, \tag{5.64}$$

where the covariant derivative implicitly acts on a propagator with colour indices (ij) as

$$\frac{D}{Dx^\mu}G^{ij}(x,y) \equiv \Big(\frac{\partial}{\partial x^\mu} + \mathrm{i}\,gA_\mu^{ij}(x)\Big)G^{ij}(x,y) \tag{5.65}$$

$$\frac{D}{Dy^\mu}G^{ij}(x,y) \equiv \Big(\frac{\partial}{\partial y^\mu} - \mathrm{i}\,gA_\mu^{ij}(y)\Big)G^{ij}(x,y) \tag{5.66}$$

(these are the only combinations that occur in Feynman diagrams). For a real scalar field, or a field charged under a background A_μ^{ij} which vanishes, we have the propagator

$$G_0(x,y) = \lim_{B^{ij}\to 0} G^{ij}(x,y) = \int_0^\infty \frac{\mathrm{d}t}{(4\pi t)^{\frac{d}{2}}}\,\mathrm{e}^{-tm_{ij}^2}\,\exp\Big[\frac{(x_\mu - y_\mu)(x^\mu - y^\mu)}{4t}\Big]\,. \tag{5.67}$$

5.4.2 Gluon propagator

The ghosts have the same propagator as the scalars, but the propagator for the gluons is more complicated because of the background field strength $F_{\mu\rho}^{ji}$ appearing in Eq. (5.40). The propagator needs to satisfy

$$\bigg(\eta_{\mu\rho}\Big(\frac{D}{Dx_\sigma}\frac{D}{Dx^\sigma} + m_{ij}^2\Big) + 2\mathrm{i}gF_{\mu\rho}^{ij}\bigg)G^{ij,\rho\nu}(x,y) = \mathrm{i}\delta_\mu^\nu\delta^d(x-y)\,. \tag{5.68}$$

To diagonalize it, we can introduce idempotent projection operators P_+ P_-, $P_\perp$ via

$$(P_\pm)_\rho{}^\sigma = \frac{\eta_{\rho\alpha}^{\parallel} \pm i\mathcal{A}_{\rho\alpha}}{2}\eta^{\alpha\sigma} \qquad\qquad (P_\perp)_\rho{}^\sigma = \eta_{\rho\alpha}^{\perp}\eta^{\alpha\sigma} \tag{5.69}$$

and then we have

$$(P_+ + P_- + P_\perp)_{\mu\nu} = \eta_{\mu\nu} \qquad\qquad (P_+ - P_-)_{\mu\nu} = \mathrm{i}\,\mathcal{A}_{\mu\nu}\,. \tag{5.70}$$

so we can rewrite Eq. (5.68) as

$$\begin{aligned}\bigg(\Big(\frac{D}{Dx_\sigma}\frac{D}{Dx^\sigma} + m_{ij}^2\Big)\eta_{\mu\rho}^{\perp} + \Big(\frac{D}{Dx_\sigma}\frac{D}{Dx^\sigma} + m_{ij}^2 + 2gB^{ij}\Big)P_{\mu\rho}^{+} \\ + \Big(\frac{D}{Dx_\sigma}\frac{D}{Dx^\sigma} + m_{ij}^2 - 2gB^{ij}\Big)P_{\mu\rho}^{-}\bigg)G^{ij,\rho\nu}(x,y) = \mathrm{i}\delta_\mu^\nu\delta^d(x-y)\,.\end{aligned} \tag{5.71}$$

There are three terms here: one is like the equation for the propagator of a charged scalar with mass m_{ij}^2, and the other two are the same except the mass term has been altered by

$m_{ij}^2 \mapsto m_{ij}^2 \pm 2gB^{ij}$. Let $G_{\pm}^{ij}(x,y)$ be the scalar propagators satisfying

$$-\Big(\frac{D}{Dx_\mu}\frac{D}{Dx^\mu} + m_{ij}^2 \pm 2gB^{ij}\Big)G_{\pm}^{ij}(x,y) = \mathrm{i}\delta^d(x-y)\,. \tag{5.72}$$

If we let $G^{ij,\sigma\alpha}$ be given by

$$G^{ij,\sigma\alpha}(x,y) = -\eta_\perp^{\sigma\alpha}G^{ij}(x,y) - P_+^{\sigma\alpha}G_+^{ij}(x,y) - P_-^{\sigma\alpha}G_-^{ij}(x,y), \tag{5.73}$$

then by the orthogonality of the projection operators, we have

$$\begin{aligned}\Big(\eta_{\mu\rho}\Big(\frac{D}{Dx_\sigma}\frac{D}{Dx^\sigma} + m_{ij}^2\Big) + 2\mathrm{i}gF_{\mu\rho}^{ij}\Big)G^{ij,\rho\nu}(x,y) &= \mathrm{i}\,(P^+)_\mu{}^\nu\delta^d(x-y) + \mathrm{i}\,(P^-)_\mu{}^\nu\delta^d(x-y)\\ &\quad + \mathrm{i}\,(\eta^\perp)_\mu{}^\nu\delta^d(x-y) &(5.74)\\ &= \mathrm{i}\,\delta_\mu^\nu\,\delta^d(x-y). &(5.75)\end{aligned}$$

Noting that m_{ij}^2 enters the propagator only via a factor of $\mathrm{e}^{-tm_{ij}^2}$ in the heat kernel Eq. (5.61), so we see that the gluon propagator is therefore given by

$$\begin{aligned}G_{\mu\nu}^{ij}(x,y) &= -\int_0^\infty \mathrm{d}t\,(\eta_{\mu\nu}^\perp + P_{\mu\nu}^+\mathrm{e}^{2gB^{ij}t} + P_{\mu\nu}^-\mathrm{e}^{-2gB^{ij}t})\mathcal{K}^{ij}(x,y;t) &(5.76)\\ &= -\int_0^\infty \mathrm{d}t\,\Big(\eta_{\mu\nu}^\perp + \eta_{\mu\nu}^\parallel\cosh(2gB^{ij}t) + \mathrm{i}\,\mathcal{A}_{\mu\nu}\sinh(2gB^{ij}t)\Big)\,\mathcal{K}^{ij}(x,y;t). &(5.77)\end{aligned}$$

We can write this in the compact form

$$G_{\mu\nu}^{ij}(x,y) = -\int_0^\infty \mathrm{d}t\,\exp(2\mathrm{i}gF^{ij}t)_{\mu\nu}\,\mathcal{K}^{ij}(x,y;t)\,, \tag{5.78}$$

which has the benefit of making it easy to contract the Lorentz indices of multiple gluon propagators:

$$G_{\mu\sigma}^{ij}(x,y)G^{k\ell,\sigma}{}_\nu(z,w) = \int_0^\infty \mathrm{d}t_i\,\exp(2\mathrm{i}g(F^{ij}t_1 + F^{k\ell}t_2))_{\mu\nu}\,\mathcal{K}^{ij}(x,y;t_1)\mathcal{K}^{k\ell}(z,w;t_2)\,. \tag{5.79}$$

The propagator for the diagonal component fields Q_μ^{ii} or for charged fields Q_μ^{ij} whose background field A_μ^{ij} vanishes is

$$G_0^{\mu\nu}(x,y) = -\eta^{\mu\nu}G_0(x,y)\,. \tag{5.80}$$

The scalar and vector propagators in Eq. (5.60) and Eq. (5.78) have the symmetries

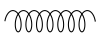
$$G^{ij}(x,y) = G^{ji}(y,x) \qquad\qquad G_{\mu\nu}^{ij}(x,y) = G_{\nu\mu}^{ji}(y,x)\,. \tag{5.81}$$

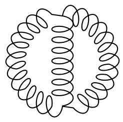
5.5 Summary of Feynman diagrams

In this section we will list all of the two-loop 1-particle-irreducible planar Feynman diagrams we get from the vertices in section 5.2. To compare easily with the string theory results, we will order the results by the colour indices, *i.e.*, we will list all of the diagrams whose propagators have the three colour indices i, j and k, say; we expect that all of these come from the field theory limit of the worldsheet whose boundaries are on the ith, jth and kth D-branes.

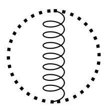
Then we should sum the following diagrams, weighted appropriately, over i, j and k, where we write

$$B_{jk} = B_1 \qquad B_{ki} = B_2 \qquad B_{ij} = B_3 \qquad m_{jk}^2 = m_1^2 \qquad m_{ki}^2 = m_2^2 \qquad m_{ij}^2 = m_3^2\,. \tag{5.82}$$

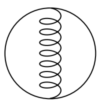
 denotes the gluon propagator, ·········· is the Faddeev-Popov ghost propagator, and ——— is the scalar field propagator.



$$= \frac{g^2}{(4\pi)^d}\int_0^\infty \frac{\prod_{i=1}^3 \mathrm{d}t_i\, \mathrm{e}^{-t_i m_i^2}}{\Delta_0^{d/2-1}\Delta_F}\Bigg\{ -\frac{3-\gamma^2}{2}\frac{(d-2)(d-3)}{\Delta_0}(t_1+t_2+t_3)$$
$$-\frac{2(d-2)}{\Delta_F}\Bigg(\frac{\sinh(gF_1t_1)}{gF_1}\Bigg(\frac{1-\gamma^2}{2}\cosh(gB_2t_2-gB_3t_3)$$
$$+\cosh(2gB_1t_1-gB_2t_2-gB_3t_3)+\text{cyclic permutations}\Bigg)\Bigg)$$
$$-\frac{2(d-2)}{\Delta_0}\Bigg(\Big(t_1+\frac{1-\gamma^2}{2}t_2\Big)\cosh(2B_2t_2-2B_3t_3)+\text{cyclic permutations}\Bigg)$$
$$-\frac{2}{\Delta_F}\Bigg(\frac{\sinh(gB_1t_1)}{gB_1}\Bigg(\cosh(2gB_1t_1-gB_2t_2-gB_3t_3)$$
$$+\frac{1-\gamma^2}{2}\cosh(3gB_3t_3-2gB_1t_1-gB_2t_2)\Bigg)+\text{cyclic permutations}\Bigg)\Bigg\} \tag{5.83}$$



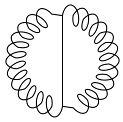
$$= \frac{1+\gamma^2}{2}\frac{g^2}{(4\pi)^d}\int_0^\infty \frac{\prod_{i=1}^3 \mathrm{d}t_i\, \mathrm{e}^{-t_i m_i^2}}{\Delta_0^{d/2-1}\Delta_F}\Bigg\{\frac{2}{\Delta_F}\frac{\sinh(F_3t_3)}{F_3}\cosh(2F_3t_3-F_1t_1-F_2t_2)$$
$$+\frac{d-2}{\Delta_0}t_3+\text{ cyclic permutations}\Bigg\} \tag{5.84}$$



$$= -N_{\mathrm{s}}\frac{g^2}{(4\pi)^d}\int_0^\infty \frac{\prod_{i=1}^3 \mathrm{d}t_i\, \mathrm{e}^{-t_i m_i^2}}{\Delta_0^{d/2-1}\Delta_F}\Bigg\{\frac{d-2}{\Delta_0}\Big(t_3+\frac{1-\gamma^2}{4}(t_1+t_2)\Big)$$
$$+\frac{2}{\Delta_F}\Bigg(\frac{\sinh(gB_3t_3)}{gB_3}\cosh(2gB_3t_3-gB_1t_1-gB_2t_2)$$

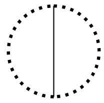
$$+\frac{1-\gamma^2}{4}\Big(\frac{\sinh(gB_1t_1)}{gB_1}\cosh(gB_3t_3-gB_2t_2) \tag{5.85}$$

$$+\frac{\sinh(gB_2t_2)}{gB_2}\cosh(gB_3t_3-gB_1t_1)\Big)\Big)+\text{ cyclic permutations}\Big\}.$$

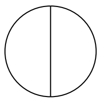


$$=\mathrm{i}\,\frac{g^2}{(4\pi)^d}\int_0^\infty\frac{\prod_{i=1}^3\mathrm{d}t_i\,\mathrm{e}^{-t_im_i^2}}{\Delta_0^{d/2-1}\Delta_F}\Big\{\Big(\frac{1+\gamma^2}{2}m_3^2-m_1^2-m_2^2\Big) \tag{5.86}$$

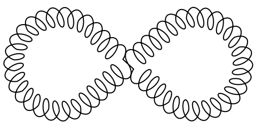
$$\times\big(d-2+2\cosh(2gB_1t_1-2gB_2t_2)\big)+\text{ cyclic permutations}\Big\}.$$



$$=-\mathrm{i}\,\frac{g^2}{(4\pi)^d}\int_0^\infty\frac{\prod_{i=1}^3\mathrm{d}t_i\,\mathrm{e}^{-t_im_i^2}}{\Delta_0^{d/2-1}\Delta_F}(m_3^2-m_1^2-m_2^2)+\text{ cyclic permutations}\,. \tag{5.87}$$



$$=\mathrm{i}\,(N_\mathrm{s}-1)\frac{3-\gamma^2}{2}(m_1^2+m_2^2+m_3^2)\frac{g^2}{(4\pi)^d}\int_0^\infty\frac{\prod_{i=1}^3\mathrm{d}t_i\,\mathrm{e}^{-t_im_i^2}}{\Delta_0^{d/2-1}\Delta_F}. \tag{5.88}$$



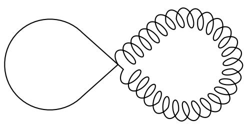
$$=\mathrm{i}\,\frac{g^2}{(4\pi)^d}\int_0^\infty\Big[\prod_{i=1}^2\frac{\mathrm{d}t_i\,\mathrm{e}^{-t_im_i^2}\,gB_i}{t_i^{d/2-1}\sinh(gB_it_i)}\Big] \tag{5.89}$$

$$\times\frac{1}{2}\Big\{d-2+2\cosh(2gB_1t_1+2gB_2t_2)$$

$$+\frac{\gamma^2-1}{2}\Big(d-2+2\cosh(2gB_1t_1-2gB_2t_2)$$

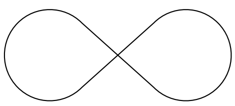
$$+\big(d-2+2\cosh(2gB_1t_1)\big)\big(d-2+2\cosh(2gB_2t_2)\big)\Big)\Big\}.$$

$$+\,\text{cyclic permutations}\,,$$



$$=\mathrm{i}\,\frac{g^2}{(4\pi)^d}\int_0^\infty\Big[\prod_{i=1}^2\frac{\mathrm{d}t_i\,\mathrm{e}^{-t_im_i^2}\,gB_i}{t_i^{d/2-1}\sinh(gB_it_i)}\Big]\frac{\gamma^2-1}{2}\big(d-2+2\cosh(2gB_2t_2)\big)N_\mathrm{S}$$

$$+\,\text{cyclic permutations}, \tag{5.90}$$



$$=\mathrm{i}\,\frac{g^2}{(4\pi)^d}\int_0^\infty\Big[\prod_{i=1}^2\frac{\mathrm{d}t_i\,\mathrm{e}^{-t_im_i^2}\,gB_i}{t_i^{d/2-1}\sinh(gB_it_i)}\Big]\big(1+\frac{\gamma^2-1}{2}(1+N_\mathrm{S})\big)N_\mathrm{S}$$

$$+\,\text{cyclic permutations}. \tag{5.91}$$

Note that the gauge choice $\gamma^2=1$ gives many of these diagrams a much simpler form, for example, the second and sixth lines of Eq. (5.83), the third and fourth lines of Eq. (5.85) and the entirety of the Eq. (5.90), the diagram with a quartic gluon-scalar vertex, vanish in this gauge. In fact, the last example is a special case of the fact that both propagators in the diagrams with quartic vertices must have the same polarization precisely when $\gamma^2 = 1$, which corresponds to both propagators coming from the same CFT in string theory.

5.5.1 Feynman diagrams

All of the diagrams are evaluated in position space. To compute amplitudes perturbatively we use the path integral. As usual, we split the Lagrangian into a quadratic part and an 'interaction' term

$$\mathcal{L} = \mathcal{L}_{QQ} + \mathcal{L}_{\phi\phi} + \mathcal{L}_{\overline{c}c} + \mathcal{L}_{\text{int}}\big[Q^{ij}_\mu, \overline{c}^{ij}, c^{ij}, \phi^{ij}_I\big] \tag{5.92}$$

and write the partition function in the presence of currents as

$$\begin{aligned} Z[J^{ij}_\mu, \eta^{ij}, \overline{\eta}^{ij}, J^{ij}_I] = \int \big[\mathcal{D}Q^{ij}_\mu\, \mathcal{D}\overline{c}^{ij}\, \mathcal{D}c^{ij}\, \mathcal{D}\phi^{ij}_I\big] \exp\Big(\mathrm{i}\int \mathrm{d}^d x\, \big(\mathcal{L}\big[Q^{ij}_\mu, \overline{c}^{ij}, c^{ij}, \phi^{ij}_I\big] \\ + J^{ij}_\mu Q^{\mu,ji} + \overline{c}^{ji}\eta^{ij} + \overline{\eta}^{ij}c^{ji} + J^{ij}_I\phi^{ji}_I\big)\Big) \end{aligned} \tag{5.93}$$

$$= \exp\Big(\mathrm{i}\,\mathcal{L}_{\text{int}}\Big[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J^{\mu,ji}}, \mathrm{i}\frac{\delta}{\delta\eta^{ji}}, \frac{1}{\mathrm{i}}\frac{\delta}{\delta\overline{\eta}^{ji}}, \frac{1}{\mathrm{i}}\frac{\delta}{\delta J^{ji}_I}\Big]\Big) Z_0[J^{ij}_\mu, \eta^{ij}, \overline{\eta}^{ij}, J^{ij}_I] \tag{5.94}$$

where

$$\begin{aligned} Z_0[J^{ij}_\mu, \eta^{ij}, \overline{\eta}^{ij}, J^{ij}_I] = \int \big[\mathcal{D}Q^{ij}_\mu\, \mathcal{D}\overline{c}^{ij}\, \mathcal{D}c^{ij}\, \mathcal{D}\phi^{ij}_I\big] \exp\Big(\mathrm{i}\int \mathrm{d}^d x\, \big(\mathcal{L}_{QQ} + \mathcal{L}_{\phi\phi} + \mathcal{L}_{\overline{c}c} \\ + J^{ij}_\mu Q^{\mu,ji} + \overline{c}^{ji}\eta^{ij} + \overline{\eta}^{ij}c^{ji} + J^{ij}_I\phi^{ji}_I\big)\Big) \end{aligned} \tag{5.95}$$

We can 'complete the square' on the scalars using the propagator which satisfies Eq. (5.64)

$$\int \mathrm{d}^d x\, \Big(\mathcal{L}_{\phi\phi} + J^{ji}_I\phi^{ij}_I\Big) = \int \mathrm{d}^d x\, \Big(-\frac{1}{2}\phi^{ji}_I(D_\mu D^\mu + m^2_{ij})\phi^{ij}_I + J^{ji}_I\phi^{ij}_I\Big) \tag{5.96}$$

$$\begin{aligned} = \int \mathrm{d}^d x\, \Big(-\frac{1}{2}\widetilde{\phi}^{ji}_I(D_\mu D^\mu + m^2_{ij})\widetilde{\phi}^{ij}_I \\ + \frac{\mathrm{i}}{2}J^{ji}_I(x)\int \mathrm{d}^d y\, G^{ij}(x,y)J^{ij}_I(y)\Big) \end{aligned} \tag{5.97}$$

where we have shifted the field to account for the current:

$$\widetilde{\phi}^{ij}_I(x) \equiv \phi^{ij}_I(x) - \mathrm{i}\int \mathrm{d}^d y\, G^{ij}(x,y)J^{ij}_I(y)\,. \tag{5.98}$$

Carrying out similar manipulations on the ghost and gluon fields, we arrive at

$$\begin{aligned} Z_0[J^{ij}_\mu, \eta^{ij}, \overline{\eta}^{ij}, J^{ij}_I] = Z_0[0,0,0,0]\, \exp\Big(-\int \mathrm{d}^d x\, \mathrm{d}^d y\, \Big[\frac{1}{2}J^{ji}_\mu(x)G^{\mu\nu,ij}(x,y)J^{ij}_\nu(y) \\ + \overline{\eta}^{ji}(x)G^{ij}(x,y)\eta^{ij}(y) + \frac{1}{2}J^{ji}_I(x)G^{ij}(x,y)J^{ij}_I(y)\Big]\Big)\,. \end{aligned} \tag{5.99}$$

This expression can be plugged into Eq. (5.94) with all the currents set to 0 to give an expression for the vacuum partition function:

$$
\begin{aligned}
Z[0,0,0,0] = Z_0[0,0,0,0] \exp\Big(\mathrm{i}\,\mathcal{L}_{\mathrm{int}}\Big[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J^{\mu,ji}}, \mathrm{i}\frac{\delta}{\delta \eta^{ji}}, \frac{1}{\mathrm{i}}\frac{\delta}{\delta \overline{\eta}^{ji}}, \frac{1}{\mathrm{i}}\frac{\delta}{\delta J_I^{ji}}\Big]\Big) \\
\times \exp\Big(-\int \mathrm{d}^d x\, \mathrm{d}^d y\, \Big[\frac{1}{2} J_\mu^{ji}(x) G^{\mu\nu,ij}(x,y) J_\nu^{ij}(y) \\
+ \overline{\eta}^{ji}(x) G^{ij}(x,y) \eta^{ij}(y) + \frac{1}{2} J_I^{ji}(x) G^{ij}(x,y) J_I^{ij}(y)\Big]\Big)\Big|_{J_\mu=\overline{\eta}=\eta=J_I=0} .
\end{aligned}
\tag{5.100}
$$

The colour indices inside the exponential are summed over; this means that charged components of the gluon and scalars are counted twice, so they enter Eq. (5.100) with the correct normalization.

We shall calculate three Feynman diagrams explicitly and state the results of the remaining ones, since they are calculated in the same way.

A Feynman diagram with D_μ vertices

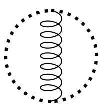
Let us begin by calculating the diagram in Eq. (5.84). The relevant interaction vertex is $\mathcal{L}_{\overline{c}cQ}$ in Eq. (5.45), which we can integrate by parts and rewrite as

$$
\begin{aligned}
\mathcal{L}_{\overline{c}cQ} = \frac{\mathrm{i}\,g}{\sqrt{2}}\big(\delta^{jk}\delta^{\ell m}\delta^{ni} - \delta^{jm}\delta^{nk}\delta^{\ell i}\big) D_\mu \overline{c}^{ij} Q^{\mu,k\ell} c^{mn} \\
- \frac{\mathrm{i}\,g\,\gamma}{\sqrt{2}}\big(\delta^{jk}\delta^{\ell m}\delta^{ni} + \delta^{jm}\delta^{nk}\delta^{\ell i}\big) \overline{c}^{ij} Q^{\mu,k\ell} D_\mu c^{mn}
\end{aligned}
\tag{5.101}
$$

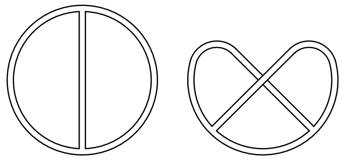
so

$$
\begin{aligned}
\mathcal{L}_{\overline{c}cQ}\Big[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J^{\mu,ji}}, \mathrm{i}\frac{\delta}{\delta \eta^{ji}}, \frac{1}{\mathrm{i}}\frac{\delta}{\delta \overline{\eta}^{ji}}\Big] = \frac{g}{\sqrt{2}}\big(\delta^{jk}\delta^{\ell m}\delta^{ni} - \delta^{i\ell}\delta^{jm}\delta^{kn}\big) D_\mu \frac{\delta}{\delta \eta^{ji}} \frac{\delta}{\delta J_\mu^{\ell k}} \frac{\delta}{\delta \overline{\eta}^{nm}} \\
- \frac{g\,\gamma}{\sqrt{2}}\big(\delta^{jk}\delta^{\ell m}\delta^{ni} + \delta^{i\ell}\delta^{jm}\delta^{kn}\big) \frac{\delta}{\delta \eta^{ji}} \frac{\delta}{\delta J_\mu^{\ell k}} D_\mu \frac{\delta}{\delta \overline{\eta}^{nm}} .
\end{aligned}
\tag{5.102}
$$

Inserting two copies of this vertex according to Eq. (5.100), we obtain



$$
\begin{aligned}
= -\frac{g^2}{4}\delta^{an}\delta^{bm}\delta^{dk}\delta^{c\ell}\delta^{fi}\delta^{ej} \int \mathrm{d}^d x\, \mathrm{d}^d y\, G^{\mu\nu,k\ell}(x,y) \\
\times \Bigg(\big(\delta^{bc}\delta^{de}\delta^{fa} - \delta^{ad}\delta^{be}\delta^{cf}\big)\big(\delta^{jk}\delta^{\ell m}\delta^{ni} - \delta^{i\ell}\delta^{jm}\delta^{kn}\big) \frac{D}{Dx^\mu} G^{ij}(x,y) \frac{D}{Dy^\nu} G^{mn}(x,y) \\
- \gamma\big(\delta^{bc}\delta^{de}\delta^{fa} - \delta^{ad}\delta^{be}\delta^{cf}\big)\big(\delta^{jk}\delta^{\ell m}\delta^{ni} + \delta^{i\ell}\delta^{jm}\delta^{kn}\big) G^{ij}(x,y) \frac{D}{Dx^\mu}\frac{D}{Dy^\nu} G^{mn}(x,y) \\
- \gamma\big(\delta^{bc}\delta^{de}\delta^{fa} + \delta^{ad}\delta^{be}\delta^{cf}\big)\big(\delta^{jk}\delta^{\ell m}\delta^{ni} - \delta^{i\ell}\delta^{jm}\delta^{kn}\big) \frac{D}{Dx^\mu}\frac{D}{Dy^\nu} G^{ij}(x,y) G^{mn}(x,y) \\
+ \gamma^2\big(\delta^{bc}\delta^{de}\delta^{fa} + \delta^{ad}\delta^{be}\delta^{cf}\big)\big(\delta^{jk}\delta^{\ell m}\delta^{ni} + \delta^{i\ell}\delta^{jm}\delta^{kn}\big) \frac{D}{Dy^\nu} G^{ij}(x,y) \frac{D}{Dx^\mu} G^{mn}(x,y)\Bigg) .
\end{aligned}
\tag{5.103}
$$



(a) (b)

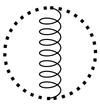
Figure 5.1: Possible double-line graph topologies with two cubic vertices.

The third and fourth lines (the ones involving double derivatives) vanish when we contract the colour indices:

$$\delta^{an}\delta^{bm}\delta^{dk}\delta^{c\ell}\delta^{fi}\delta^{ej}\left(\delta^{bc}\delta^{de}\delta^{fa} \pm \delta^{ad}\delta^{be}\delta^{cf}\right)\left(\delta^{jk}\delta^{\ell m}\delta^{ni} \mp \delta^{i\ell}\delta^{jm}\delta^{kn}\right) = 0\,. \tag{5.104}$$

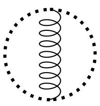
The second and fifth lines contain contributions from the planar (Fig. 5.1a) and non-planar (Fig. 5.1b) diagrams, where in Fig. 5.1 we have used double-line notation to indicate the colour structure. The non-planar diagram arises when all colour indices are contracted with each other, i.e. from terms containing $\delta^{an}\delta^{ni}\delta^{if}\delta^{fc}\delta^{c\ell}\delta^{\ell m}\delta^{mb}\delta^{be}\delta^{ej}\delta^{jk}\delta^{kd}\delta^{da}$ or $\delta^{an}\delta^{nk}\delta^{kd}\delta^{de}\delta^{ej}\delta^{jm}\delta^{mb}\delta^{bc}\delta^{c\ell}\delta^{\ell i}\delta^{if}\delta^{fa}$. We are not interested in these diagrams because every propagator is uncharged with respect to the background field, and therefore these diagrams don't contribute to the effective action. We don't expect to obtain these diagrams from string theory since our calculation begins with a planar worldsheet.

The planar diagram is given by



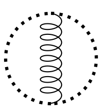
$$= -\frac{g^2}{4}\left(\delta^{jk}\delta^{\ell m}\delta^{ni} + \delta^{i\ell}\delta^{jm}\delta^{kn}\right)\int \mathrm{d}^d x\, \mathrm{d}^d y\, G^{\mu\nu,k\ell}(x,y) \tag{5.105}$$
$$\times\left(\frac{D}{Dx^\mu}G^{ij}(x,y)\frac{D}{Dy^\nu}G^{mn}(x,y) + \gamma^2\frac{D}{Dy^\nu}G^{ij}(x,y)\frac{D}{Dx^\mu}G^{mn}(x,y)\right).$$

This can be written as two copies of the same term differing only in the order of the colour indices:



$$= -g^2\,\frac{1+\gamma^2}{4}\int \mathrm{d}^d x\, \mathrm{d}^d y\, \frac{D}{Dx^\mu}G^{ij}(x,y)\frac{D}{Dy^\nu}G^{jk}(x,y)G^{\mu\nu,ki}(x,y)$$
$$+\big((ijk) \leftrightarrow (kji)\big)\,. \tag{5.106}$$

Inserting the expressions for the scalar and gluon propagators in Eq. (5.60) and Eq. (5.78), we obtain



$$= g^2\,\frac{1+\gamma^2}{4}\int_0^\infty \mathrm{d}t_i\, \exp(2\mathrm{i}gF^{ki}t_3)^{\mu\nu}$$
$$\times\int \mathrm{d}^d x\, \mathrm{d}^d y\, \frac{D}{Dx^\mu}\mathcal{K}_{ij}(x,y;t_1)\,\frac{D}{Dy^\nu}\mathcal{K}_{jk}(x,y;t_2)\,\mathcal{K}_{ki}(x,y;t_3) \tag{5.107}$$
$$+\big((ijk) \leftrightarrow (kji)\big)\,.$$

We need to carry out the integral over x and y in the second line. First of all, we can calculate

$$\frac{D}{Dx^\mu}\mathcal{K}_{ij}(x,y;t_1) = \frac{\beta(F^{ij}; t_1)_{\mu\rho} + \mathrm{i}\,F^{ij}_{\mu\rho}}{2}(x^\rho - y^\rho)\,\mathcal{K}_{ij}(x,y;t_1) \tag{5.108}$$

$$\frac{D}{Dy^\mu}\mathcal{K}_{jk}(x,y;t_2) = -\frac{\beta(F^{jk}; t_2)_{\nu\sigma} - \mathrm{i}\,F^{jk}_{\nu\sigma}}{2}(x^\sigma - y^\sigma)\,\mathcal{K}_{jk}(x,y;t_2) \tag{5.109}$$

where $\beta(F,t)$ is defined in Eq. (5.62). We then have

$$\begin{aligned}
&\frac{D}{Dx^\mu}\mathcal{K}_1(x,y;t_1)\,\frac{D}{Dy^\nu}\mathcal{K}_2(x,y;t_2)\,\mathcal{K}_3(x,y;t_3) \\
&\quad = -\frac{\beta(F^1; t_1)_{\mu\rho} + \mathrm{i}\,F^1_{\mu\rho}}{2}\,\frac{\beta(F^2; t_2)_{\nu\sigma} - \mathrm{i}\,F^2_{\nu\sigma}}{2}(x^\rho - y^\rho)(x^\sigma - y^\sigma)\prod_{i=1}^{3}\mathcal{K}_i(x,y;t_i)\,,
\end{aligned} \tag{5.110}$$

where we've simplified the notation for F_{ij}, B_{ij} and m^2_{ij} by writing $1,2,3$ in place of the colour indices $(ij),(jk),(ki)$, respectively. Now, the complex phase in $\prod_{i=1}^3 \mathcal{K}_i(x,y;t_i)$ vanishes due to the fact that $\sum_{i=1}^3 B_i = 0$, and we have

$$\prod_{i=1}^{3}\mathcal{K}_{A_i}(x,y;t_i) = \mathrm{e}^{\frac{1}{4}(x^\mu - y^\mu)\mathcal{B}(F^i; t_i)_{\mu\nu}(x^\nu - y^\nu)}\prod_{i=1}^{3}\frac{\mathrm{e}^{-t_i m_i^2}}{(4\pi t_i)^{\frac{d}{2}}}\frac{gB_i t_i}{\sinh(gB_i t_i)}. \tag{5.111}$$

where $\mathcal{B}(F^i; t_i)_{\mu\nu} = \sum_{i=1}^3 \beta(F^i; t_i)_{\mu\nu}$. Note that the integrand is a function of $z = x - y$ so we can replace the integral over x with an integral over z while the integral over y will gives a factor corresponding to the volume of spacetime, which we won't write explicitly. We get

$$\begin{aligned}
&\int \mathrm{d}^d x\,\mathrm{d}^d y\,(x^\rho - y^\rho)(x^\sigma - y^\sigma)\prod_{i=1}^{3}\mathcal{K}_i(x,y;t_i) \\
&\qquad = \prod_{i=1}^{3}\frac{\mathrm{e}^{-t_i m_i^2}}{(4\pi t_i)^{\frac{d}{2}}}\frac{gB_i t_i}{\sinh(gB_i t_i)}\int \mathrm{d}^d z\, z^\rho z^\sigma \mathrm{e}^{\frac{1}{4}z^\mu \mathcal{B}(F^i; t_i)_{\mu\nu} z^\nu}.
\end{aligned} \tag{5.112}$$

This is a moment of a Gaussian integral so it can be written as

$$\int d^d z\; z^\rho z^\sigma \mathrm{e}^{\frac{1}{4}z^\mu \mathcal{B}(F^i; t_i)_{\mu\nu} z^\nu} = -2(\mathcal{B}(F^i; t_i)^{-1})^{\rho\sigma}\int d^d z\; \mathrm{e}^{\frac{1}{4}z^\mu \mathcal{B}(F^i; t_i)_{\mu\nu} z^\nu}. \tag{5.113}$$

To make is easier to contract the Lorentz indices, it is a good idea to write $\mathcal{B}$ as

$$\mathcal{B}(F^i, t_i)_{\mu\nu} = \Delta_0 \prod_{i=1}^{3}\frac{1}{t_i}\,\eta^\perp_{\mu\nu} + \Delta_F \prod_{i=1}^{3}\frac{gB_i}{\sinh(gB_i t_i)}\,\eta^\parallel_{\mu\nu} \tag{5.114}$$

where Δ_0 and Δ_F are defined in Eq. (4.67) and Eq. (4.65) respectively, and $\eta^\perp_{\mu\nu}$ and $\eta^\parallel_{\mu\nu}$ are defined in Eq. (5.63). Eq. (5.114) holds because $\sum_{i=1}^3 B_i = 0$. Note that $\mathcal{B}(F^i, t_i)_{\mu\nu}$ is diagonal so its inverse is simply

$$(\mathcal{B}(F^i,\,t_i)^{-1})^{\rho\sigma} = \Delta_0^{-1}\prod_{i=1}^{3} t_i\,\eta_\perp^{\rho\sigma} + \Delta_F^{-1}\prod_{i=1}^{3}\frac{\sinh(gB_it_i)}{gB_i}\,\eta_\parallel^{\rho\sigma}\,. \tag{5.115}$$

Noting that we can write

$$(\beta(F^1,t_1)_{\mu\rho} + \mathrm{i}\,F^1_{\mu\rho}) = \frac{\eta^\perp_{\mu\rho}}{t_1} + \frac{gB_1}{\sinh(gB_1t_1)}\Big(\eta^\parallel_{\mu\rho}\cosh(gB_1t_1) + \mathrm{i}\,\mathcal{A}_{\mu\rho}\sinh(gB_1t_1)\Big) \tag{5.116}$$

$$(\beta(F^2,t_2)_{\nu\sigma} - \mathrm{i}\,F^2_{\nu\sigma}) = \frac{\eta^\perp_{\nu\sigma}}{t_2} + \frac{gB_2}{\sinh(gB_2t_2)}\Big(\eta^\parallel_{\nu\sigma}\cosh(gB_2t_2) - \mathrm{i}\,\mathcal{A}_{\nu\sigma}\sinh(gB_2t_2)\Big) \tag{5.117}$$

where $\mathcal{A}_{\mu\nu}$ is defined in Eq. (5.57), we can use the tensor contractions

$$\eta_\perp^{\rho\sigma}\eta^\perp_{\mu\rho}\,\eta^\perp_{\nu\sigma} = \eta^\perp_{\mu\nu} \qquad \eta_\parallel^{\rho\sigma}\eta^\parallel_{\mu\rho}\,\eta^\parallel_{\nu\sigma} = \eta^\parallel_{\mu\nu} \qquad \eta_\parallel^{\rho\sigma}\mathcal{A}_{\mu\rho}\,\eta^\parallel_{\nu\sigma} = \mathcal{A}_{\mu\nu}$$

$$\eta_\parallel^{\rho\sigma}\eta^\parallel_{\mu\rho}\,\mathcal{A}_{\nu\sigma} = -\mathcal{A}_{\mu\nu} \qquad \eta_\parallel^{\rho\sigma}\mathcal{A}_{\mu\rho}\,\mathcal{A}_{\nu\sigma} = \eta^\parallel_{\mu\nu} \tag{5.118}$$

to get

$$\begin{aligned}&(\mathcal{B}(F^i,t_i)^{-1})^{\rho\sigma}\,(\beta(F^1,t_1)_{\mu\rho} + \mathrm{i}\,F^1_{\mu\rho})\,(\beta(F^2,t_2)_{\nu\sigma} - \mathrm{i}\,F^2_{\nu\sigma}) \\ &\quad = \Delta_0^{-1}t_3\,\eta^\perp_{\mu\nu} + \Delta_F^{-1}\frac{\sinh(gB_3t_3)}{gB_3}\Big(\eta^\parallel_{\mu\nu}\cosh(gB_1t_1{+}gB_2t_2) + \mathrm{i}\,\mathcal{A}_{\mu\nu}\sinh(gB_1t_1{+}gB_2t_2)\Big)\,.\end{aligned} \tag{5.119}$$

This can be written as

$$\begin{aligned}&(\mathcal{B}(F^i,t_i)^{-1})^{\rho\sigma}(\beta(F^1,t_1)_{\mu\rho} + \mathrm{i}\,F^1_{\mu\rho})\,(\beta(F^2,t_2)_{\nu\sigma} - \mathrm{i}\,F^2_{\nu\sigma}) \\ &\qquad = \boldsymbol{\Delta}^{-1}_{\mu\sigma}\,\mathcal{S}(F_3,t_3)^{\sigma\rho}\,\exp(\mathrm{i}\,(gF_1t_1 + gF_2t_2))_{\rho\nu}\end{aligned} \tag{5.120}$$

where

$$\boldsymbol{\Delta}^{-1}_{\mu\sigma} = \Delta_0^{-1}\,\eta^\perp_{\mu\sigma} + \Delta_F^{-1}\,\eta^\parallel_{\mu\rho} \qquad \mathcal{S}(F_3,t_3)^{\sigma\rho} = t_3\,\eta_\perp^{\sigma\rho} + \frac{\sinh(gB_3t_3)}{gB_3}\,\eta_\parallel^{\sigma\rho}. \tag{5.121}$$

Finally, we use a Wick rotation to evaluate

$$\int \mathrm{d}^dz\,\mathrm{e}^{\frac{1}{4}z^\mu\mathcal{B}(F^i,t_i)_{\mu\nu}z^\nu} = -\mathrm{i}\,(4\pi)^{\frac{d}{2}}(\det\mathcal{B})^{-\frac{1}{2}} \tag{5.122}$$

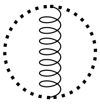
$$= -\mathrm{i}\,(4\pi)^{\frac{d}{2}}\Delta_0^{1-\frac{d}{2}}\Delta_F^{-1}\prod_{i=1}^{3}\frac{\sinh(gB_it_i)}{gB_it_i}t_i^{d/2}\,, \tag{5.123}$$

where we've used the expression for $\mathcal{B}$ in Eq. (5.114). Putting Eq. (5.120) and Eq. (5.123) together, we arrive at

$$\int \mathrm{d}^dx\,\mathrm{d}^dy\,\frac{D}{Dx^\mu}\,\mathcal{K}_1(x,y;t_1)\frac{D}{Dy^\nu}\,\mathcal{K}_2(x,y;t_2)\,\mathcal{K}_3(x,y;t_3) \tag{5.124}$$

$$= -\mathrm{i}\,\frac{1}{(4\pi)^d}\,\frac{\prod_{i=1}^{3}\mathrm{e}^{-t_i m_i^2}}{\Delta_0^{d/2-1}\Delta_F}\,\boldsymbol{\Delta}^{-1}_{\mu\sigma}\,\mathcal{S}(F_3,t_3)^{\sigma\rho}\,\exp(\mathrm{i}(gF_1t_1+gF_2t_2))_{\rho\nu},$$

which is also useful for evaluating the Feynman diagrams in Eq. (5.83) and Eq. (5.85). Plugging Eq. (5.124) into Eq. (5.107) and contracting the Lorentz indices and reinstating the colour indices, we get

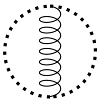


$$= -\mathrm{i}\,\frac{g^2}{(4\pi)^d}\,\frac{1+\gamma^2}{4}\int_0^\infty \mathrm{d}t_i\,\frac{\mathrm{e}^{-t_1m_{ij}^2-t_2m_{jk}^2-t_3m_{ki}^2}}{\Delta_0^{d/2-1}\Delta_F}\Big[\frac{d-2}{\Delta_0}t_3 + \frac{2}{\Delta_F}\frac{\sinh(gB_{ki}t_3)}{gB_{ki}}\cosh(2gB_{ki}t_3-gB_{ij}t_1-gB_{jk}t_2)\Big] + \big((ijk)\leftrightarrow(kji)\big)\,. \tag{5.125}$$

Now, swapping $\big((ijk)\leftrightarrow(kji)\big)$ is equivalent to rewriting

$$B_{ki}\to -B_{ki} \qquad B_{ij}\leftrightarrow -B_{jk} \qquad m_{ki}^2\to m_{ki}^2 \qquad m_{ij}^2\leftrightarrow m_{jk}^2 \tag{5.126}$$

which is a symmetry of the integral in Eq. (5.125) (if we rename the integration variables $t_1\leftrightarrow t_2$). Therefore[1]



$$= -\mathrm{i}\,\frac{g^2}{(4\pi)^d}\,\frac{1+\gamma^2}{2}\int_0^\infty \mathrm{d}t_i\,\frac{\mathrm{e}^{-t_1m_{ij}^2-t_2m_{jk}^2-t_3m_{ki}^2}}{\Delta_0^{d/2-1}\Delta_F}\Big[\frac{d-2}{\Delta_0}t_3 + \frac{2}{\Delta_F}\frac{\sinh(gB_{ki}t_3)}{gB_{ki}}\cosh(2gB_{ki}t_3-gB_{ij}t_1-gB_{jk}t_2)\Big]\,. \tag{5.127}$$

To get the full contribution to the QFT from diagrams of this topology, we need to sum over the different possible colour structures, which is equivalent to summing over cyclic permutations of the B_i's.

A Feynman diagram with m_{ij} vertices

The Feynman diagrams with the same topology as the one in Eq. (5.84) but with an odd number of scalar propagators instead of an odd number of gluon propagators can be computed similarly, but they are simpler since the vertices do not involve covariant derivatives but are proportional to the scalar VEVs m_{ij}. For example, let us calculate the Feynman diagram in Eq. (5.86).

The relevant vertex is in Eq. (5.46); if we relabel the indices using $m_I^{kj}+m_I^{ji}+m_I^{ik}=0$, we have

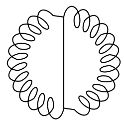
$$\mathcal{L}_{Q^2\phi} = \frac{g}{\sqrt{2}}\big((1+\gamma)m_I^{kj}-(1-\gamma)m_I^{ik}\big)\phi^{ij}Q_\mu^{jk}Q^{\mu,ki}\,, \tag{5.128}$$

Using two copies of

$$\mathcal{L}_{\Phi QQ}\Big[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J},\frac{1}{\mathrm{i}}\frac{\delta}{\delta J^\mu}\Big] = \mathrm{i}\,\frac{g}{\sqrt{2}}\big((1-\gamma)m_I^{ik}-(1+\gamma)m_I^{kj}\big)\frac{\delta}{\delta J_I^{ji}}\,\frac{\delta}{\delta J^{kj,\mu}}\,\frac{\delta}{\delta J_\mu^{ik}}\,, \tag{5.129}$$

[1]

according to Eq. (5.100) and contracting the colour indices, we arrive at

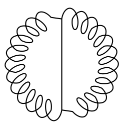


$$= \frac{g^2}{4}\big((1-\gamma)m_I^{kj} - (1+\gamma)m_I^{ik}\big)\big((1-\gamma)m_I^{ik} - (1+\gamma)m_I^{kj}\big)$$
$$\times \int \mathrm{d}^d x\, \mathrm{d}^d y\, G^{ij}(x,y)\, G^{jk}_{\mu\nu}(x,y)\, G^{\mu\nu,ki}(x,y)\,, \tag{5.130}$$

plus a non-planar part, where we've used $m_I^{jk} = -m_I^{kj}$ and $m_I^{ki} = -m_I^{ik}$. Inserting the identity

$$\big((1-\gamma)a - (1+\gamma)b\big)\big((1-\gamma)b - (1+\gamma)a\big) = (1+\gamma^2)(a+b)^2 - 2(a^2+b^2) \tag{5.131}$$

with $m_I^{jk} + m_I^{ki} = -m_I^{ij}$ and using Eq. (5.36) to rewrite the expression in terms of the bare squared masses of the fields, and then inserting the expressions for the scalar and gluon propagators in Eq. (5.60) and Eq. (5.78), we obtain

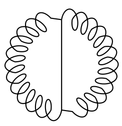


$$= \frac{g^2}{4}\big((1+\gamma^2)m_{ij}^2 - 2(m_{jk}^2 + m_{ki}^2)\big) \int \mathrm{d}t_i\, \exp(2\mathrm{i}gF_{jk}t_1)_{\mu\nu} \exp(2\mathrm{i}gF_{ki}t_2)^{\mu\nu}$$
$$\times \int \mathrm{d}^d x\, \mathrm{d}^d y\, \mathcal{K}_{ij}(x,y;t_3)\mathcal{K}_{jk}(x,y;t_1)\mathcal{K}_{ki}(x,y;t_2). \tag{5.132}$$

Rewriting the second line using Eq. (5.111) and carrying out the integration over x and y with Eq. (5.123), and writing

$$\exp(2\mathrm{i}gF_{jk}t_1)_{\mu\nu} \exp(2\mathrm{i}gF_{ki}t_2)^{\mu\nu} = d - 2 + 2\cosh(2gB^{jk}t_1 - 2gB^{ki}t_2)\,, \tag{5.133}$$

(where the relative sign between B_{jk} and B_{ki} comes from transposing one of the Hermitian matrices before contracting the Lorentz indices), we end up with



$$= -\frac{\mathrm{i}}{(4\pi)^d}\frac{g^2}{4}\big((1+\gamma^2)m_{ij}^2 - 2(m_{jk}^2 + m_{ki}^2)\big) \tag{5.134}$$
$$\times \int_0^\infty \mathrm{d}t_i\, \frac{\mathrm{e}^{-t_1 m_{jk}^2 - t_2 m_{ki}^2 - t_3 m_{ij}^2}}{\Delta_0^{\frac{d}{2}-1}\Delta_F}\big(d - 2 + 2\cosh(2gB^{jk}t_1 - 2gB^{ki}t_2)\big)\,.$$

This diagram corresponds only to a particular colour structure; to find the full contribution to the QFT coming from diagrams with this field content we need to sum over all cyclic permutations of the B_i's and multiply by a factor of 2 from counting the two possible orientations of the propagators. The diagrams in Eq. (5.87) and Eq. (5.88) can be calculated similarly to this one.

A Feynman diagram with a quartic vertex

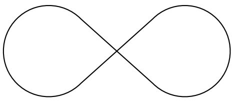
Finally, we will calculate a diagram with a quartic vertex. Let us calculate the diagram in Eq. (5.91). The relevant interaction term in the Lagrangian comes from Eq. (5.51) and can be rewritten as

$$\mathcal{L}_{\phi^4} = \frac{g^2}{4}(\delta_{KI}\delta_{LJ} + (\gamma^2 - 1)\delta_{IL}\delta_{JK})\phi^{ij}_K\phi^{jk}_L\phi^{k\ell}_I\phi^{\ell i}_J \tag{5.135}$$

so if we insert

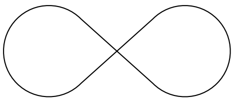
$$\mathcal{L}_{\phi^4}\Big[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J^{ji}_I}\Big] = \frac{g^2}{4}(\delta_{KI}\delta_{LJ} + (\gamma^2 - 1)\delta_{IL}\delta_{JK})\frac{\delta}{\delta J^{ji}_K}\frac{\delta}{\delta J^{kj}_L}\frac{\delta}{\delta J^{\ell k}_I}\frac{\delta}{\delta J^{i\ell}_J} \tag{5.136}$$

in Eq. (5.100), we obtain the diagram



$$= \mathrm{i}\,\frac{g^2}{4}(\delta_{KI}\delta_{LJ} + (\gamma^2 - 1)\delta_{IL}\delta_{JK})\int d^dx\,\big(G^{\ell i}(x,x)G^{jk}(x,x)\delta_{ik}\delta_{IJ}\delta_{LK} + G^{k\ell}(x,x)G^{\ell i}(x,x)\delta_{\ell j}\delta_{LI}\delta_{JK}\big)\,, \tag{5.137}$$

plus a non-planar term. Relabelling the indices as $(jk\ell) \to (abc)$ in the first term and $(k\ell i) \to (abc)$ in the second term and contracting the flavour indices, then inserting the expression for the scalar propagator in Eq. (5.60) and dropping, as usual, the factor of the volume of spacetime coming from the integral over x, we get



$$= \mathrm{i}\,\frac{g^2}{2}\Big(1 + \frac{\gamma^2 - 1}{2}(1 + N_\mathrm{s})\Big)N_\mathrm{s} \times \frac{1}{(4\pi)^d}\int_0^\infty \prod_{i=1}^{2}\Big[\frac{\mathrm{d}t_i}{t_i^{d/2-1}}\frac{gB_i\mathrm{e}^{-t_i m_i^2}}{\sinh(gB_it_i)}\Big] \tag{5.138}$$

where $(B_1, m_1^2) \equiv (B^{ab}, m_{ab}^2)$ and $(B_2, m_2^2) \equiv (B^{bc}, m_{bc}^2)$. The diagrams in Eq. (5.89) and Eq. (5.90) can be calculated similarly.

5.6 Comparison between QFT and string theory

It is clear by inspecting equations Eq. (4.91) to Eq. (4.104) for the QFT limit of the string amplitude calculated in the symmetric p_i parametrization and equations Eq. (4.129) to Eq. (4.132) for the QFT limit of the remaining terms in the string amplitude that we obtain all of the 1PI two-loop QFT Feynman diagrams Eq. (5.83) to Eq. (5.91) in the gauge $\gamma^2 = 1$, as well as some unaccounted-for terms (to precise the second line of Eq. (4.129)) which have the expected form of 1PR diagrams completely factorized into two independent loop integrals.

The terms in the p_i QFT limit in Eq. (4.91) to Eq. (4.104) clearly correspond to the Feynman diagrams with the topology of Fig. 4.4a; we can identify each term in Eq. (4.91) with a particular Feynman diagram. $\mathbf{f}_\perp^{111}$ in Eq. (4.96) matches the first line of Eq. (5.83) in which all three propagators are polarized perpendicular to the magnetic field; $\mathbf{f}_\parallel^{001}\mathbf{f}_\perp^{110}$ plus its cyclic permutations matches the third line of Eq. (5.83) in which one propagator is polarized parallel and two are polarized perpendicular to the magnetic fields, and so on.

The situation for the remainder of the QFT limit of the string amplitude in Eq. (4.129) is not as obvious because there are two possible Feynman diagram topologies for the various terms to map onto. In fact, we see that the three terms $\mathbf{f}_{\|}^{11}$, $\mathbf{f}_{\perp}^{11}$ and $\mathbf{f}_{\rm scal}^{11}$ given in the first line of Eq. (4.133) correspond to the Feynman diagrams with the topology of Fig. 4.4c in the gauge $\gamma^2 = 1$; $\mathbf{f}_{\|}^{11} + \mathbf{f}_{\perp}^{11}$ are mapped to Eq. (5.89) while $\mathbf{f}_{\rm scal}^{11}$ gives the diagram with two scalar propagators in Eq. (5.91). The diagram with one scalar propagator and one gluon propagator in Eq. (5.90) vanishes for $\gamma^2 = 1$ so we should be unsurprised that there is nothing corresponding to it on the string theory side. There are several compelling reasons to believe this: firstly that the QFT Feynman diagram in Eq. (5.89) contains the structure $2\cosh(2gB_1t_1 + 2gB_2t_2)$ and the terms in the first line of Eq. (4.133) are the only place on the string theory side where such an object arises. Secondly, this is the only way on the string theory side that we could explain the fact that both propagators in Eq. (5.89) and Eq. (5.91) have to have the same polarization for $\gamma^2 = 1$.

The remaining terms in Eq. (4.129) coming from the product $\big(\mathbf{f}_{\|}^{10} + \mathbf{f}_{\perp}^{10} + \mathbf{f}_{\rm scal}^{10} + \mathbf{f}_{\rm gh}^{10}\big)\big(\mathbf{f}_{\|}^{01} + \mathbf{f}_{\perp}^{01} + \mathbf{f}_{\rm scal}^{01} + \mathbf{f}_{\rm gh}^{01}\big)$ have the wrong structure to match any of our 1PI diagrams for $\gamma^2 = 1$ because they have only two propagators, each of which can be polarized independently. In the factorized form of the second line of Eq. (4.133), it is clear that this term can be written it in terms of a trace over the propagators:

$$
\begin{aligned}
&- g^2 \prod_{i=1}^{2} \left[\frac{1}{(4\pi)^{d/2}} \int_0^\infty \frac{\mathrm{d}t_i}{t_i^{d/2-1}} \frac{\mathrm{e}^{-t_i m_i^2}\, gB_i}{\sinh(gB_it_i)} \left(\mathbf{f}_{\|}^{i} + \mathbf{f}_{\perp}^{i} + \mathbf{f}_{\rm scal}^{i} + \mathbf{f}_{\rm gh}^{i} \right) \right] \\
&\qquad\qquad = -g^2 \prod_{i=1}^{2} \left[\eta^{\mu\nu} G^i_{\mu\nu}(x,x) + N_{\rm s} G^i(x,x) - 2G^i(x,x) \right] ,
\end{aligned}
\tag{5.139}
$$

where the vector and scalar propagators are given in Eq. (5.78) and Eq. (5.60). It seems like these terms should correspond to 1PR Feynman diagrams with the topology of Fig. 4.4b.

We can characterize the 1PI QFT diagrams according to the origins of the terms in the infinite products in the string theory amplitude. All of the terms on the string theory side which correspond to 1PI QFT diagrams originate from the square root of the muliplier of a single Schottky group element in the expansion of the infinite products in Eq. (4.9). The terms in Eq. (4.91) come from ${k_1}^{\frac{1}{2}}$, ${k_2}^{\frac{1}{2}}$ or $k(\mathbf{S}_1^{-1}\mathbf{S}_2)^{\frac{1}{2}}$ and the terms in the first line of Eq. (4.133) come from $k(\mathbf{S}_1\mathbf{S}_2)^{\frac{1}{2}}$. These correspond to only one single term being taken from the infinite products in Eq. (4.9). On the other hand, the terms which seem to correspond to the 1PR QFT diagrams come from the product of two terms arising from two Schottky group elements in the infinite products, e.g. ${k_1}^{\frac{1}{2}}$ from one factor and ${k_2}^{\frac{1}{2}}$ from another factor.

Recall from Fig. 4.3 that $k(\mathbf{S}_1\mathbf{S}_2)^{\frac{1}{2}}$, which is the source of the terms in the first line of Eq. (4.133), corresponds to a homology cycle which passes around both handles crossing itself in the middle, and $\mathbf{S}_1\mathbf{S}_2$ is the only Schottky group element with this property which survives in the QFT limit; it is also the only one which contributes to the Feynman diagrams with quartic vertices.

Chapter 6

Effective actions

6.1 Quantum effective actions and low-energy effective actions

The *quantum action*, or 1PI effective action Γ (which many authors just call the effective action) is an object we can define for quantum fields theories which has the property that the tree level Feynman graphs we obtain from it give the complete scattering amplitude [104]. It can be written as the integral of an *effective potential* $\Gamma = \int \mathrm{d}^d x\, V$. V has the property of being the ground state average energy density as a function of some order parameter, such that a vacuum state of the theory should realise a minimum of V [105, 106]. In the standard analogy between quantum field theory and statistical mechanics, $-\Gamma$ corresponds to the Gibbs free energy [105, 107].

The effective action is useful for studying theories with spontaneously broken symmetries, *i.e.*, theories whose lagrangian $\mathcal{L}$ has a symmetry which is not a symmetry of the vacuum [108]. Use of the effective action allows us to survey all possible vacua of a theory simultaneously, as opposed to perturbing about a chosen vacuum. Since radiative (quantum) corrections are included in Γ, we can potentially find vacua which are not minima of the classical action [106].

In general in a gauge-theory, the effective action is a gauge-dependent quantity [109]. However, physical observables such as physical masses and coupling constants and S-matrix elements computed from it are independent of gauge parameters [110]. Since Γ is associated with the energy density of the vacuum at a stationary point, it is important that it is gauge-independent at such points, and indeed this is the case [105].

6.2 The Callan-Symanzik β function for scalar QED

Scalar QED is a quantum field theory whose field content consists of a charged scalar field minimally coupled to a U(1) gauge field. It is similar in some senses to (spinor) QED, but all of the fields are bosonic so it can be studied with only the NS sector of an open type II superstring. We can build a model from a U(2) theory on the Coulomb branch, using non-coincident parallel D_3 branes as in Fig. 4.1, but with only two of them.

The Callan-Symanzik β function is a physical quantity which describes how the renormalized coupling constant varies with the characteristic energy scale of the physical process in question. In some cases, it can be calculated using the background field method in the following way. One takes a renormalizable theory and inserts by hand a background field and calculates the effective action. The expression will contain divergences, but these are not physical divergences, rather, they arise from the fact that the variables our Lagrangian is expressed in terms of (in our case, the mass m and the overall normalization B of the background field) are not the appropriate physical variables. Because the theory is renormalizable, we can rescale the 'bare' quantities m and B such that the divergences disappear. This has to be done order-by-order in perturbation theory.

When using the background field method to calculate the effective action and the β function, only the background field B undergoes wavefunction renormalization; the quantum fields do not renormalize, which is one of the main attractions of this method [111, 112].

We start by naïvely writing down the unrenormalized h-loop effective actions $\mathcal{L}^{(h)}$ as simply the sum of all h-loop Feynman diagrams, and then the unrenormalized 2-loop effective action is simply the sum $\sum_{i=0}^{2} \mathcal{L}^{(2)}$:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(0)}[B] + \mathcal{L}^{(1)}[gB, m^2] + \mathcal{L}^{(2)}[gB, m^2] + \mathcal{O}(\alpha^3)\,, \tag{6.1}$$

where here $\alpha \equiv \frac{g^2}{4\pi}$ is the fine structure constant. $\mathcal{L}^{(1)}$ can be written as a sum of a finite part $\mathcal{L}_R^{(1)}[gB, m^2]$, a regularized part proportional to $\mathcal{L}^{(0)}[B]$, and another 'cosmological constant' term independent of the background field. $\mathcal{L}^{(2)}[gB, m^2]$ can be written as a sum of a finite part, a part proportional to $\mathcal{L}^{(0)}[gB]$, and a part proportional to $\frac{\partial \mathcal{L}^{(1)}}{\partial m^2}[gB, m^2]$.

To understand the appearance of the derivative with respect to m^2, we observe that m^2 must not be the physical mass of the theory, the physical mass is $m_R^2 = m^2 + \delta m^2$. At the order we've calculated, δm^2 is of order α; when we include more loops then δm^2 will also include correction of higher power in α. Then the physical effective action will be equal to

$$\begin{aligned}
\mathcal{L}_{\text{eff}} &= \mathcal{L}^{(0)}[B_R] + \mathcal{L}_R^{(1)}[g_R B_R, m_R^2] + \mathcal{L}_R^{(2)}[g_R B_R, m_R^2] + \mathcal{O}(\alpha^3) && (6.2)\\
&= \mathcal{L}^{(0)}[B_R] + \mathcal{L}_R^{(1)}[g_R B_R, m^2] + \delta m^2 \frac{\partial \mathcal{L}^{(1)}}{\partial m^2}[g_R B_R, m^2] \\
&\qquad + \mathcal{L}_R^{(2)}[g_R B_R, m^2] + \mathcal{O}(\alpha^3) && (6.3)
\end{aligned}$$

where we have Taylor expanded the $\mathcal{L}^{(i)}$s about the bare mass. This only makes a difference to $\mathcal{L}^{(1)}$: $\mathcal{L}^{(0)}$ is independent of the mass, and the correction to $\mathcal{L}^{(2)}$ would be of order α^3, and we are only computing the effective action up to order α^2, so we have put $\mathcal{L}_R^{(2)}[g_R B_R, m_R^2] = \mathcal{L}_R^{(2)}[g_R B_R, m^2] + \mathcal{O}(\alpha^3)$.

We can calculate the renormalized field strength B_R by requiring that $\mathcal{L}_R^{(0)}[B_R]$ has the same form as $\mathcal{L}^{(0)}[B]$, i.e. $\mathcal{L}_R^{(0)}[B_R] = -\frac{1}{2}B_R^2$, where $\mathcal{L}_R^{(0)}$ is the sum of the contributions at each loop level which are proportional to $\mathcal{L}^{(0)}[B]$. This defines the renormalization constant Z_3 via $B_R = B Z_3^{-\frac{1}{2}}$. In order for $\mathcal{L}_R^{(1)}$ and $\mathcal{L}_R^{(2)}$ not to change due to this field redefinition, it is also necessary that we renormalize the coupling as $g_R = gZ_3^{\frac{1}{2}}$.

As computed this way, Z_3 depends on the bare mass m^2 which we used to regularize the integrals, but since m^2 itself is not a physical quantity but is given by $m_R^2 - \delta m^2$, we need to insert this expression into Z_3, and only then will we be able to correctly calculate the β function.

To be able to carry out this calculation with the use of the string theory techniques carried out above, we should calculate the relevant Feynman diagrams in the non-linear Gervais-Neveu gauge [11].

Similarly to the calculation in section 5.2, let us write the components of the gauge field and the scalar field as

$$\mathscr{A}_\mu = A_\mu + Q_\mu = A_\mu^3 T^3 + Q_\mu^3 T^3 + Q_\mu^0 T^0 \,; \tag{6.4}$$

$$\Phi = \langle \Phi \rangle + \Phi' = \frac{m}{g} T^3 + \phi T^+ + \phi^* T^- \,, \tag{6.5}$$

where we have taken advantage of the fact that the gauge group is now U(2) to express the fields in terms of Pauli matrices as:

$$T^0 = \frac{1}{2}\mathbf{1} \qquad T^i = \frac{1}{2}\sigma^i \qquad T^\pm = \frac{1}{\sqrt{2}}(T^1 + \mathrm{i}\,T^2) \tag{6.6}$$

which satisfy

$$\mathrm{Tr}(T^A T^B) = \frac{1}{2}\delta^{AB} \qquad \mathrm{Tr}(T^\pm T^\pm) = 0 \qquad \mathrm{Tr}(T^\pm T^\mp) = \frac{1}{2} \tag{6.7}$$

$$[T^0, \cdot] = 0 \qquad [T^\pm, T^\mp] = \pm T^3 \qquad [T^3, T^\pm] = \pm T^\pm \tag{6.8}$$

$$\{T^0, x\} = x \quad \{T^\pm, T^\pm\} = 0 \quad \{T^\pm, T^\mp\} = T^0 \quad \{T^\pm, T^3\} = 0 \quad \{T^3, T^3\} = T^0 \,. \tag{6.9}$$

Since our goal in this section is to model scalar QED, we have not explicitly included the full field content that would arise naturally from putting U(2) Yang-Mills theory on the Coulomb branch, *i.e.* we haven't explicitly included the charged (with respect to the background field) and massive off-diagonal components of the gauge field or the uncharged diagonal components of the scalar.

The interaction vertices can be derived as they were in section 5.2 by taking the usual Lagrangian for dimensionally-reduced Yang-Mills theory, expanding it in components, and then discarding all terms involving fields which don't appear in our scalar QED model.

In background Feynman gauge $D_\mu^{\mathscr{A}} Q^\mu = 0$, the contribution from the the Q^0 term would vanish since it only appears in the Lagrangian via a commutator, which vanishes. In Gervais-Neveu gauge, however, a term including Q^2 appears in the gauge fixing Lagrangian which means Q^0 has to be accounted for.

Let us first calculate the one-loop correction to the effective action. It is no harder to carry out the general case for a broken U(N) theory instead of a broken U(2) theory, so let us be general.

The classical action is given by

$$\mathcal{L}_{\rm cla} = -\frac{1}{2}\mathrm{Tr}\big(F_{\mu\nu}F^{\mu\nu}\big)\,. \tag{6.10}$$

The classical gauge field A_μ only interacts with other fields via a commutator in the covariant derivative, so it is convenient to split it into a part proportional to the identity $A_\mu^{\mathrm{U}(1)}$ which doesn't interact with anything, and a traceless part $A_\mu^{\mathrm{SU}(N)}$:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,; \qquad\qquad A_\mu(x) = A_\mu^{\mathrm{U}(1)} + A_\mu^{\mathrm{SU}(N)}\,; \tag{6.11}$$

and we can write $A_\mu^{\mathrm{SU}(N)}$ as

$$A_\mu^{\mathrm{SU}(N)} = \frac{1}{\sqrt{2}}x_1\eta_{\mu 2}\begin{pmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_N \end{pmatrix}\,; \qquad\qquad \sum_{i=1}^{N} B_i = 0\,, \tag{6.12}$$

where the B_i are normalized as U(1) gauge fields. Splitting F up into

$$F_{\mu\nu}^{\mathrm{U}(1)} = \partial_\mu A_\nu^{\mathrm{U}(1)} - \partial_\nu A_\mu^{\mathrm{U}(1)}\,; \qquad\qquad F_{\mu\nu}^{\mathrm{SU}(N)} = \partial_\mu A_\nu^{\mathrm{SU}(N)} - \partial_\nu A_\mu^{\mathrm{SU}(N)}\,, \tag{6.13}$$

we see that $F_{\mu\nu}^{\mathrm{U}(1)}$ is also proportional to the identity in the gauge group while $F_{\mu\nu}^{\mathrm{SU}(N)}$ is also traceless, so we get

$$\mathcal{L}_{\rm cla} = -\frac{1}{2}\mathrm{Tr}\big(F_{\mu\nu}^{\mathrm{U}(1)}F^{\mathrm{U}(1),\mu\nu}\big) - \frac{1}{2}\mathrm{Tr}\big(F_{\mu\nu}^{\mathrm{SU}(N)}F^{\mathrm{SU}(N),\mu\nu}\big)\,, \tag{6.14}$$

where the cross-term has cancelled because it is the trace of the product of a term proportional to the identity and a traceless term.

Now, we can use the identity

$$\sum_{i=1}^{N} B_i^2 = \frac{1}{N}\Big(\big(\sum_{i=1}^{N} B_i\big)^2 + \sum_{j=2}^{N}\sum_{i=1}^{j-1}(B_i - B_j)^2\Big) \tag{6.15}$$

along with the fact that $\sum_{i=1}^{N} B_i = 0$ to rewrite the second term in Eq. (6.14) as

$$-\frac{1}{2}\mathrm{Tr}\big(F_{\mu\nu}^{\mathrm{SU}(N)}F^{\mathrm{SU}(N),\mu\nu}\big) = -\frac{1}{N}\sum_{j=2}^{N}\sum_{i=1}^{j-1} B_{ij}^2\,, \tag{6.16}$$

where we've defined

$$B_{ij} \equiv \frac{B_i - B_j}{\sqrt{2}}\,. \tag{6.17}$$

We can take equation (6.16) to be our tree-level effective action. We want to check that this is normalized properly to be compared to the SU(2) case. There, we have written the background field in terms of the Pauli matrix T^3 via

$$A_\mu^{\mathrm{U}(2)} = x_1\eta_{\mu 2}BT^3 \qquad\qquad T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.18}$$

or, writing this in the form of equation (6.12), we have

$$A_\mu^{\mathrm{U}(2)} = x_1\eta_{\mu 2}\frac{1}{\sqrt{2}}\begin{pmatrix} B/\sqrt{2} & 0 \\ 0 & -B/\sqrt{2} \end{pmatrix} \tag{6.19}$$

so to match the normalization we are using, we need $B_1 = B/\sqrt{2}$ and $B_2 = -B/\sqrt{2}$ so $B_{12} = B$ and the tree-level Lagrangian is given in terms of the background field as

$$-\frac{1}{2}\mathrm{Tr}\big(F_{\mu\nu}^{\mathrm{SU}(2)}F^{\mathrm{SU}(2),\mu\nu}\big) = -\frac{1}{2}B^2\,. \tag{6.20}$$

Now we want to compute the one-loop effective action. Generically, it can be found from the expression [107]

$$\Gamma_1 = \frac{\mathrm{i}}{2}\mathrm{Tr}\log\big[-\frac{\delta^2\mathcal{L}_1}{\delta\phi\,\delta\phi}\big]\,. \tag{6.21}$$

For the scalars, the quadratic part of the Lagrangian is given (after a partial integration) by

$$\mathcal{L} = -\frac{1}{2}\sum_{i,j=1}^{N}\phi^{ij}(x)\big(D_\mu D^\mu + m_{ij}^2\big)\phi^{ji}(x) \tag{6.22}$$

Where, since the background field is diagonal, the component-wise covariant derivative acts as

$$D_\mu\phi^{ij} = \partial_\mu\phi^{ij} + \mathrm{i}gB_{ij}x_1\eta_{\mu 2}\phi^{ij}\,. \tag{6.23}$$

It's the same as the covariant derivative for a complex scalar field in the U(1) background gauge field $B_{ij}x_1\eta_{\mu 2}$, so the propagator can be written down straight away from Eq. (5.60). Using the heat kernel defined in the same equation, we can see straight away that

$$\frac{d}{d(m_{ij}^2)}(-t)^{-1}\mathcal{K}_{ij}(x,y;t) = \mathcal{K}_{ij}(x,y;t) \tag{6.24}$$

from which it follows that

$$\log(D_\mu^{ij}D^{\mu,ij} + m_{ij}^2) = -\mathrm{i}\int_0^\infty \frac{\mathrm{d}t}{t}\,\mathcal{K}_{ij}(x,y;t)\,, \tag{6.25}$$

plus a constant. Substituting this in to equation (6.21), we arrive at an expression for the unrenormalized one-loop effective action:

$$\Gamma_1 = \frac{1}{2}\frac{1}{(4\pi)^{\frac{d}{2}}}\sum_{i,j=1}^{N}\int \mathrm{d}^d x \int_0^\infty \frac{\mathrm{d}t}{t^{\frac{d}{2}}}\,\frac{gB_{ij}\mathrm{e}^{-tm_{ij}^2}}{\sinh(gB_{ij}t)} \tag{6.26}$$

The integral is divergent but we can make a Taylor-series expansion of the integrand in terms of the integration variable t and subtract the two divergent terms, using

$$\frac{gB_{ij}}{\sinh(gB_{ij}t)} = \frac{1}{t} - \frac{g^2B_{ij}^2 t}{6} + \mathcal{O}(t^3)\,. \tag{6.27}$$

We will simply subtract the $\frac{1}{t^{\frac{d}{2}+1}}$ pole, equating it to a cosmological constant which isn't important for investigation of the background field, while the $\frac{1}{t^{\frac{d}{2}-1}}$ pole contributes to the wavefunction renormalization in spacetime dimension $d=4$, which we now fix. Therefore we can write

$$\begin{aligned}\Gamma_1 = &\frac{1}{2}\frac{1}{(4\pi)^2}\sum_{i,j=1}^{N}\int \mathrm{d}^d x \int_0^\infty \frac{\mathrm{d}t}{t^2}\,\mathrm{e}^{-tm_{ij}^2}\left(\frac{gB_{ij}}{\sinh(gB_{ij}t)} - \frac{1}{t} + \frac{g^2B_{ij}^2 t}{6}\right)\\ &+ \frac{1}{2}\frac{1}{(4\pi)^2}\sum_{i,j=1}^{N}\int \mathrm{d}^d x \int_0^\infty \frac{\mathrm{d}t}{t^3}\,\mathrm{e}^{-tm_{ij}^2}\\ &- \frac{1}{2}\frac{g^2}{(4\pi)^2}\sum_{i,j=1}^{N}\frac{B_{ij}^2}{6}\int \mathrm{d}^d x \int_0^\infty \frac{\mathrm{d}t}{t}\,\mathrm{e}^{-tm_{ij}^2}\,.\end{aligned} \tag{6.28}$$

For $N=2$, the first term is normalized the same as the 1 loop scalar QED effective action in equation (10) of [113] and equation (3.51) of [114] since the only nonzero contributions come from B_{12} and B_{21}, cancelling the overall factor of $\frac{1}{2}$. The second line of Eq. (6.28) is just the B_{ij}-independent cosmological constant.

Now, we can use the fact that the integrand is symmetric under swapping $(ij) \leftrightarrow (ji)$ to re-express the sum, and also we regulate the divergent $\mathrm{d}t$ integral by inserting a proper-time cutoff, then we find that the last line of Eq. (6.28) can be written as

$$\frac{g^2}{(4\pi)^2}\sum_{j=2}^{N}\sum_{i=1}^{j-1}\log(t_0 m_{ij}^2\mathrm{e}^{\gamma_{\mathrm{E}}})\frac{B_{ij}^2}{6}\int \mathrm{d}^d x\,. \tag{6.29}$$

where γ_{E} is the Euler-Mascheroni constant. Note that term-by-term, it is proportional to the tree-level effective action in equation (6.16), *i.e.* the classical action for our given background. According to the philosophy of renormalization, we should absorb these divergences into redefinitions of the wavefunction by requiring that the tree level effective action (*i.e.* just the classical action) as a function of the renormalized background field strengths B_{ij}^R is equal to the same function of the 'bare' background field strengths B_{ij} plus the the divergent terms, *i.e.* we need

$$\Gamma_0[B_{ij}^R] = \Gamma_0[B_{ij}] + \frac{g^2}{(4\pi)^2}\sum_{j=2}^{N}\sum_{i=1}^{j-1}\log(t_0 m_{ij}^2\mathrm{e}^{\gamma_{\mathrm{E}}})\frac{B_{ij}^2}{6}\int \mathrm{d}^d x \tag{6.30}$$

$$= -\frac{1}{N}\sum_{j=2}^{N}\sum_{i=1}^{j-1} B_{ij}^2\big(1-\frac{N}{6}\frac{g^2}{(4\pi)^2}\log(t_0 m_{ij}^2 \mathrm{e}^{\gamma_{\mathrm{E}}})\big)\int \mathrm{d}^d x \tag{6.31}$$

which holds if the renormalized field strengths are related to the bare field strengths via

$$(B_{ij}^R)^2 = \big(1-\frac{N}{6}\frac{g^2}{(4\pi)^2}\log(t_0 m_{ij}^2 \mathrm{e}^{\gamma_{\mathrm{E}}})\big)B_{ij}^2\,. \tag{6.32}$$

In the background field method, this is all we need to know in order to calculate the β function. Noting that the form of

$$\Gamma_1^R = \frac{1}{2}\frac{1}{(4\pi)^2}\sum_{i,j=1}^{N}\int \mathrm{d}^d x\int_0^\infty \frac{\mathrm{d}t}{t^2}\,\mathrm{e}^{-tm_{ij}^2}\left(\frac{gB_{ij}}{\sinh(gB_{ij}t)}-\frac{1}{t}+\frac{g^2B_{ij}^2 t}{6}\right) \tag{6.33}$$

is unchanged if we make the substitution

$$B_{ij}\to B_{ij}^R \qquad\qquad g\to g_R \equiv \big(1-\frac{N}{6}\frac{g^2}{(4\pi)^2}\log(t_0 m_{ij}^2 \mathrm{e}^{\gamma_{\mathrm{E}}})\big)^{-\frac{1}{2}}g \equiv \sqrt{Z}g\,, \tag{6.34}$$

which defines the renormalization coefficient Z. In terms of the fine structure constant $\alpha = \frac{g^2}{4\pi}$, we have

$$\alpha_R = Z\alpha_0 \qquad\qquad Z^{-1} = 1+\frac{N}{6}\frac{\alpha_0}{4\pi}\log\big(\frac{(m_{ij}^2 t_0)^{-1}}{\mathrm{e}^{\gamma_{\mathrm{E}}}}\big)\,. \tag{6.35}$$

where α_0 and α_R are the bare and renormalized values of α, respectively. The β function can then be calculated from the equation $\beta_\alpha(\alpha_R) = -\frac{\partial\alpha_R}{\partial\log m}\Big|_{t_0}$ from which we find

$$\beta_\alpha(\alpha_R) = \frac{N}{3}\frac{\alpha_R^2}{4\pi}\,. \tag{6.36}$$

The β function is also commonly expressed not in terms of the fine-structure constant α but in terms of the coupling constant g; since the β function does not describe a scalar field but it is the coefficient of a vector field, we can transform this via

$$\beta_g(g) = \frac{1}{\partial\alpha/\partial g}\beta_\alpha(\alpha(g)) = \frac{N}{2}\frac{g^3}{48\pi^2}\,; \tag{6.37}$$

for $N=2$ this matches the scalar QED β function β_e on p. 470 of [107].

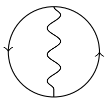
We have checked, therefore, that our setup is correctly normalized; we can move on to use it to calculate the two-loop correction to the scalar QED β function.

6.2.1 Ritus' calculation for (scalar) QED

The two loop-correction to the β-function was calculated by Ritus for QED in [115] and for scalar QED in [113]. The general idea is to consider a charged scalar or fermion in a constant background U(1) gauge field and find the tree-level propagator as an exact function of the background field strength B. This allows the effective action to be written down at two loops as an expansion in 'vacuum' Feynman graphs, where the background field enters via the propagator. Some of the diagrams need to be regularized, and it is found as expected that the divergences are all proportional to the one-loop effective action, and that they can therefore be made to vanish if the gauge coupling g and squared-mass m^2 of the matter field are renormalized in a certain way.

Let us calculate all of the two loop Feynman diagrams contributing to the scalar QED effective action, in the Gervais-Neveu gauge (using the relevant terms of the action in section 5.2 that involve only the fields we are interested in).

The diagram with two scalar propagators and one photon propagator becomes slightly modified from Ritus' calculation in [113] due to the additional interaction vertices in the Lagrangian coming from the Gervais-Neveu gauge fixing; the diagram now depends on the gauge parameter γ:



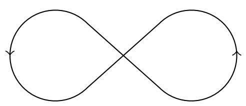
$$= -\mathrm{i}\frac{g^2}{(4\pi)^d}\iiint_0^\infty \mathrm{d}t\,\mathrm{d}t'\,\mathrm{d}s\,\mathrm{e}^{-(t+t')m^2}\frac{\Delta_F^{-1}\Delta_0^{1-\frac{d}{2}}}{\cosh(gBt)\cosh(gBt')}\left[\frac{d-2}{\Delta_0}s+\frac{2\cosh(gB(t-t'))s}{\Delta_F\cosh(gBt)\cosh(gBt')}\right] + \mathrm{i}\frac{1-\gamma^2}{2}\frac{g^2}{(4\pi)^d}\iint_0^\infty \frac{dt_1}{t_1^{\frac{d}{2}}}\frac{dt_2}{t_2^{\frac{d}{2}}}\frac{gBt_1\mathrm{e}^{-m^2t_1}}{\sinh(Bt_1)}\frac{gBt_2\mathrm{e}^{-m^2t_2}}{\sinh(gBt_2)} \tag{6.38}$$

where

$$\Delta_F = \frac{\sinh(gB(t_1+t_2))}{gB}t_3 + \frac{\sinh(gBt_1)}{gB}\frac{\sinh(gBt_2)}{gB} \tag{6.39}$$

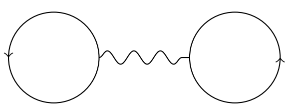
$$\Delta_0 = \lim_{B\to 0}\Delta_F = (t_1t_3+t_2t_3+t_1t_2)\,. \tag{6.40}$$

Recalling that we imposed the Gervais-Neveu gauge condition *before* dimensionally reducing and obtained a new quartic scalar vertex, we get a Feynman diagram with a figure-of-eight topology that is not found in the standard gauge in [113]:



$$= \mathrm{i}\frac{\gamma^2 g^2}{(4\pi)^d}\iint_0^\infty \frac{\mathrm{d}t_1}{t_1^{\frac{d}{2}}}\frac{\mathrm{d}t_2}{t_2^{\frac{d}{2}}}\frac{gBt_1\mathrm{e}^{-m^2t_1}}{\sinh(gBt_1)}\frac{gBt_2\mathrm{e}^{-m^2t_2}}{\sinh(gBt_2)}. \tag{6.41}$$

Note that this diagram is actually proportional to the second term (proportional to $(1-\gamma^2)$) in Eq. (6.38). It doesn't completely cancel the γ-dependence, however. In fact, there is a new type of connected, but not 1PI, diagram which appears in Gervais-Neveu gauge: the one-particle-reducible (1PR) diagram

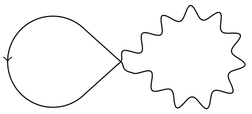


$$= -\frac{1}{2}\mathrm{i}\frac{\alpha^2 g^2}{(4\pi)^d}\iint_0^\infty \frac{\mathrm{d}t_1}{t_1^{\frac{d}{2}}}\frac{\mathrm{d}t_2}{t_2^{\frac{d}{2}}}\frac{gBt_1\mathrm{e}^{-t_1m^2}}{\sinh(gBt_1)}\frac{gBt_2\mathrm{e}^{-t_2m^2}}{\sinh(gBt_2)}. \tag{6.42}$$

which vanishes when $\gamma = 0$.

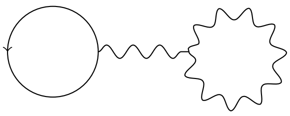
It is a good consistency check to note that when Eq. (6.38), Eq. (6.41) and Eq. (6.42) are added, the dependence on γ cancels, and we get the same result as Ritus' calculation which comes only from the diagram with the same topology as Eq. (6.38).

Feynman diagrams with only one charged propagator also contribute: the following vanishes, in fact, in Gervais-Neveu gauge $\gamma^2 = 1$



$$= \mathrm{i}(1-\gamma^2)d\frac{g^2}{(4\pi)^d}\iint_0^\infty \frac{\mathrm{d}t_1}{t_1^{\frac{d}{2}}}\frac{\mathrm{d}t_2}{t_2^{\frac{d}{2}}}\frac{gBt_1\mathrm{e}^{-m^2t_1}}{\sinh(gBt_1)}\,. \tag{6.43}$$

In fact, the gauge-dependence of Eq. (6.43) exactly cancels that of a 1PR diagram with one charged propagator, *i.e.*



$$= \mathrm{i}d\frac{\gamma^2 g^2}{(4\pi)^d}\iint_0^\infty \frac{dt_1}{t_1^{\frac{d}{2}}}\frac{dt_2}{t_2^{\frac{d}{2}}}\frac{gBt_2\mathrm{e}^{-t_2m^2}}{\sinh(gBt_2)}, \tag{6.44}$$

which also didn't appear in [113].

Since these diagrams all sum to give exactly the same result as the sum of the diagrams in the gauge used by Ritus in [113], the rest of the calculation goes through identically and the two-loop correction to the scalar QED β-function can be correctly recovered.

There is a caveat, namely, that to show exact correspondence between the two calculations, we have had to use 1PR diagrams where we were supposed to be computing a 1PI effective action.

It can be shown explicitly with a reproduction of Ritus' long calculation using only the 1PI diagrams with an explicit Gervais-Neveu gauge parameter γ^2 that the β function is actually independent of γ^2, and therefore the 1PR diagrams are superfluous, as expected.

It has been shown in chapters 4.3 and 5 that the 1PI vacuum diagrams in a gauge theory in the appropriate non-linear gauge can be found, sector-by-sector, by systematically isolating the appropriate terms in a Schottky-group expansion of the corresponding string theory vacuum diagram. We have just seen that by isolating appropriate sectors of an $N = 2$ Yang-Mills theory on the Coulomb branch in the Gervais-Neveu gauge, we can find the two loop correction to the scalar QED β function. It follows that it would be possible to carry out the calculation directly from string theory and obtain the physically correct answer, by manually selecting only those factors in the integrand corresponding to the field content of scalar QED.

Chapter 7

Outlook

We have seen that in the super-Schottky parametrization of super moduli space, we can find precisely a correspondence between terms in the expansion of the two loop vacuum amplitude, and individual Feynman diagrams, in the non-linear gauge Eq. (5.27). This matching holds not only in the pure Yang-Mills theory, but also in a dimensionally-reduced version coupled to scalars. Moreover, the matching holds even when the scalar fields are given VEVs, corresponding to separating the D-branes on which the open strings are ending.

Using the fact that the correspondence between string theory and QFT holds not only at the level of amplitudes but also sector-by-sector on the worldsheet, we have been able to isolate only certain fields we are interested in for certain applications, allowing us to obtain, for example, the Callan-Symanzik β function for scalar QED at two loops by selecting only the appropriate massless fields, finding agreement with the expression in the literature.

In forthcoming work, we will calculate the two-loop β function of the full theory we have considered, of a dimensionally-reduced Yang-Mills theory with scalar VEVs. This is more complicated than the scalar QED case, because there are many more diagrams involved, and because the various fields can undergo *a priori* different mass-renormalizations, which means the renormalization techniques used in [115, 113] have to be applied with more care.

The spacetime theory we have been considering, a dimensionally-reduced version of bosonic Yang-Mills theory, is not the full low-energy theory of type IIB superstrings; the full low-energy theory in $d = 4$ is $\mathcal{N} = 4$ super-Yang-Mills coupled to Einstein gravity. This suggests two obvious directions in which this work can proceed. First of all, the Ramond sector of the open string theory should be incorporated into our procedure for finding diagram-by-diagram correspondences, since the Ramond sector of the open string corresponds to spacetime fermions and is therefore necessary for calculations in super-Yang-Mills theory, or, indeed, even for obtaining simple models like (spinor) QED or physically interesting models like Yang-Mills coupled to fermionic matter.

It is technically complicated to generalize our calculations to incorporate the Ramond sector; the reason for this is that the super-projective transformations that are used to build SRSs with super-Schottky groups are geometrically equivalent to sewing pairs of Neveu-Schwarz punctures; Ramond punctures come from a different type of singularity in the super-conformal structure of the worldsheet [60, 79] so more work is necessary.

Another obvious direction in which the calculations can be generalized is to include the gravitational sector of the low energy theory by calculating *closed* string vacuum amplitudes. We could consider, for example, graviton vacuum amplitudes analogous to the gluon vacuum amplitudes we've discussed here, but we could also study graviton scattering amplitudes. Indeed, while the double-annulus worldsheet we've been most interested in has a natural interpretation near one boundary of super-moduli space as a two-loop open string vacuum diagram, we can also investigate the same worldsheet topology through the closed string channels, in which it corresponds to a tree-level three-point function of closed string states being emitted (or absorbed) by D-branes. The physical quantity under investigation is then the gravitational interaction between three D-branes. This description is useful near the boundary of moduli space in which our Schottky group multipliers are close to 1 instead of 0; in this region the series we use are not convergent and we need to switch to a different description, for example, to represent the worldsheet by a different Schottky group generated by our b_i cycles instead of our a_i cycles. Much of the technology we've used is still useful: for example, the D-branes can be given velocities—useful for investigating gravitational interactions—by giving analytically continuing monodromies of the worldsheet fields $\epsilon_\mu \to \mathrm{i}\epsilon_\mu$. In the case of interactions between D_0-branes, this setup has been investigated in the literature in the $\alpha' \to 0$ limit around the two complementary boundaries of moduli space [116] but no full string derivation is known.

Everything we have calculated (except from the example in section 2.4) has been in terms of 'vacuum' diagrams without external states (although they are not true vacuum diagrams since the open strings or quantum fields in question have been coupled to background gauge fields *via* a modified propagator). A natural and important extension of the work will be to include external states, so that physical scattering amplitudes can be found. It is possible that the Schottky group techniques we've employed could lead to interesting simplifications of multi-loop QFT scattering amplitudes.

Appendix A

Conventions

We use the metric with signature $(-,+,\ldots,+)$ for string theory calculations and $(+,-,\ldots,-)$ for quantum field theory calculations.

Our mode expansion of ∂X^μ is given by Eq. (2.132) which is different from that in *e.g.* [62] and [21] since our formula has an overall factor of $\sqrt{2\alpha'}$ where theirs has an overall factor of $\left(\frac{\alpha'}{2}\right)^{\frac{1}{2}}$. Their expressions for *e.g.* OPE's between chiral fields can therefore be translated into our language by making the substitution $\alpha' \mapsto 4\alpha'$.

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